Promoting Critical Mathematics Literacy

in Secondary Mathematics Teacher Education

By

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Dedication

We are convinced that adherence to the values of nonviolence will usher in a more peaceful, civilized world order in which more effective and fair governance, respectful of human dignity and the sanctity of life itself, may become a reality.

In implementing the principles of this Charter we call upon all to work together towards a just, killing-free world in which everyone has the right not to be killed and responsibility not to kill others.

All states, institutions and individuals must support efforts to address the inequalities in the distribution of economic resources, and resolve gross inequities which create a fertile ground for violence. The imbalance in living conditions inevitably leads to lack of opportunity and, in many cases, loss of hope.

Civil society, including human rights defenders, peace and environmental activists must be recognized and protected as essential to building a nonviolent world as all governments must serve the needs of their people, not the reverse. Conditions should be created to enable and encourage civil society participation, especially that of women, in political processes at the global, regional, national and local levels.

To address all forms of violence we encourage scientific research in the fields of human interaction and dialogue, and we invite participation from the academic, scientific and religious communities to aid us in the transition to nonviolent, and nonkilling societies.

(Permanent Secretariat of the World Summit of Nobel Peace Laureates, 2007)

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Abstract

This study examines how critical mathematical literacy teachers conceptualize their practices and how those practices were demonstrated in the classroom. Practices were considered from an ontology of mathematics education, specific to critical mathematical literacy, in which classroom interactions question what it means to do mathematics as an individual, as a citizen, as a community member. The study adopted an ethnographic study of three participants who self-identified, or were nominated by colleagues or peers, as critical mathematics secondary school teachers. The emphasis focused on understanding the core values and beliefs of these teachers to better comprehend how this subculture conceptualizes critical mathematics literacy (CML) practices. Analysis took particular note of the nature of strategies used for CML instruction, as well as means for developing an understanding of the meaning of participants' classroom actions. Critical mathematical literacy was interpreted as six domains of mathematical understanding that prepare students to explore their life situations through an understanding of how mathematical comprehension of social and moral decisions can lead to transformative social change.

A Discussion Concerning the Ontology of the Critical

Mathematical Literacy Classroom

Introduction

How many students can parse a newspaper advertisement offering home equity loans at 14.25%? Is that a good rate; is it reasonable to expect students to understand this context? Should this even be something we want our students to use mathematics to think about? How about using spatial reasoning to figure out a route on a subway map; or, being able to understand a bus schedule? Imagine what mathematics teaching would be like if it engaged students to see how mathematical thinking can be used in everyday issues. That is to say, what if mathematics teaching taught children how there are mathematics outside the classroom and gave them the mathematical confidence and tools to think independently, ask intelligent questions of society, and be motivated to change things for the better. Imagine the classroom life of such students and their teacher. Now believe that this mathematics teacher is you.

In searching for what it means to be a critical mathematical literacy (CML) teacher, this project incorporates several perspectives on the education of mathematics teachers, ways the mathematics teacher develops professional knowledge, and the role of a mathematics teacher educator. In considering a perspective on instruction, this work draws heavily of the early work of Peirce (1903) who commented that mathematical knowledge "must, if it is to be properly grounded, aim...to discover not how things actually are, but how they [are] supposed to be" (p. 121). This distinction, in positioning mathematical knowledge as how things are supposed to be encountered in everyday experience, moves away from assumptions of mathematical knowledge as propositional (A. Peressini & Peressini, 2007) to mathematics that is less absolutist.

Mathematics educators must move past a Platonic representation of mathematical knowledge as a preexisting ideal, to a more global system in which one can explore things as they are situated in common, lived societal and cultural experiences. That is to say, if there is a shift in foundational thinking, how individuals, mathematics teachers, are thinking, what is society going to value as mathematics teachers; what kind of knowledge? Should the curriculum be more traditional and formalist or can one begin to accept it as having historical value, and thus included, but also address everyday experiences that will give children more common-sense understanding. Once one begins to interpret mathematical knowledge in contrast to traditional Platonism, the teaching of mathematics must not only be capable of interacting with changing society and cultural experiences, but also must be seen as incapable of entirely interpreting what is to be taught as certain; that is, pedagogy must now move away from seeing mathematical knowledge as skills and mastery in equations of numbers of pizza slices or speeds of trains traveling toward each other from opposite directions, to an ideology in which mathematical knowledge and pedagogy become susceptible to critical discourse, public analysis, and debate.

Reflecting on the historical role and production of mathematics knowledge in relation to education, educators see a primary interest in meeting the needs of a technocratic society. For example, as documents such as *A Nation at Risk* (Denning, 1983) emphasized preparing students with mathematical skills for scientific and technical careers and, in turn, preparing mathematics teachers as predominately addressing classical ideas and concepts. Society must enable teachers to prepare their students to participate in modern society, rather than overlooking relations between mathematics and the liberal arts, social and political issues, and urban existence. Increased competition for public funding in teacher preparation privileges monoculture practices (Lincoln & Cannella, 2004) and narrow scholarship (Cannella & Lincoln, 2009) focused on the production of quantitative results, citing often conflicting results of “What Works” (Schoenfeld, 2006; Viadero, 2004), and evidence-based research as more credible than critical pedagogical inquiry.

Also lacking in the preparation of mathematics teachers are ways teachers may develop their students' mathematical confidence. By this, I am referring to guiding teachers to understand why their students' need to feel comfortable to use classical and everyday mathematical knowledge in ways that are culturally and personally fulfilling. That is, one may see the role of preparing mathematics teachers as an opportunity to prepare thoughtful mathematics educators with lessons for mathematical activity meaningful to their students' lives and for active participation in society.

To address these issues, mathematics teacher education should emphasize the mathematics teacher's generation of pedagogical knowledge rather than simply seeking justifications for teaching in one particular style. That is to say, in emphasizing pedagogical knowledge over justification, encourages practices and means for critique and critical discourse and teaching from and through a diversity of styles while also recognizing that the same is true of students. We want to move past the expectation of algorithms and procedures that limit students to short answers (Schoenfeld, 1988) to teaching in which expansion of the answers, through peer or class discussion is valued. Fostering such a perspective views the mathematics teacher, and students, as constantly cycling in a system of knowledge generation in which beliefs, facts, experiences, and concepts pass through realms of the individual ("I conjecture that the sum of three angles within a triangle is 180 degrees") to public ("As a group, we have measured and added all angles in four different triangles to agree that perhaps it is the case that triangles' angles sum to 180 degrees"). The point is not that comparison and discussion of a limited number of examples establishes knowledge but that social negotiation of meaning contributes to individual's mathematical knowledge and allows for the generation of new considerations ("I wonder if there is a triangle whose angles do not sum to 180 degrees"). This perspective draws on social constructivist ideas that consider how the individual's and the public's understanding of "mathematics each contributes to the creation and re-creation of the other" (Ernest, 1991, p. 43) and aims to facilitate critical thought, the questioning of autocratically transmitted mathematical knowledge, and the acknowledgment of only that which has been sensibly explained through public classroom discourse. Such teacher preparation values the teacher as a "transformative intellectual" (Giroux & McLaren, 1986, p. 301) who prepares students to become positive agents of change. This transformative teacher questions the means through which knowledge is produced, making outcomes of education meaningful, critical, and ultimately emancipatory (Aronowitz & Giroux, 1985; Skovsmose, 1994) against a continuation of present gender (Boaler, 2002), race (Martin, 2000; Moses & Cobb, 2001), and socioeconomic and cultural (Hinchey, 2010; Skovsmose & Valero, 2001; Valero, 2004) inequities that more traditional classrooms maintain.

Teachers must challenge students to communicate their ideas in a professional and friendly classroom manner, yet incorporate their previous knowledge and experience, which are made public through classroom dialogue—"a process of learning and knowing … with the objective of dismantling oppressive structures and mechanisms prevalent both in education and society" (Freire & Macedo, 1995). Through this process, educators will consider what culturally relevant mathematics practices look like and determine ways to build on the common practices students already bring to the classroom. Engaging teachers with discussion about the means and ways to include sociopolitical issues in their mathematics instruction is a part of mathematics-education research where there is little information. That is to say while prior studies have considered the application of socio-mathematical issues among middle school students (Gutstein, 2003) , underrepresented populations (Clinchy, 1996; Nasir, Hand, & Taylor, 2008; Stinson, 2006), vocational and returning adult students (Frankenstein, 1990), research which folds in teachers' awareness of sociopolitical and economic realities to respect and acknowledge the plurality of student cultures and histories, from a critical (Freire, 1970; Kincheloe, 2008; Tate, 1995) and humanizing (Bartolome, 1994) mathematical sense, has not previously been fully explored. Such an exploration will consider how incorporating critical and humanizing pedagogy in ways emphasizing the social, political, economic, and cultural roles of mathematics in challenging existing professional and mathematical understanding.

Finally, this study is steeped in the social use of mathematics and its related pedagogy. As a mathematics educator, that this mode of instruction engages both teacher and student to understand the ways in which mathematics interacts with the modern world, using realistic social problems in which mathematics can be situated. For example, in the context of a lesson on proportion and probability, one might consider using authentic police data to investigate important social questions such as, "Given the data, what are the probabilities, if you are a person who is African-American, Hispanic, or Caucasian, that your stopped car will be searched by police?" and "Based on these findings, what statements might we make about possible discrimination based on an individual's ethnicity?" This work of the critical mathematics teacher "begins with the recognition of the suffering of the oppressed in our local communities and around the planet" (Kincheloe, 2008, p. 140) and begins to open the education of students to the narratives, actions, and opportunities for positive social action by replacing absolutist views of mathematics with one that mathematics is, at its core, fallible. That is to say, in composing and understanding mathematical statements in a fallibilist perspective there is more to the notion of what becomes true and truth than simply what was historically thought of as solid certainty. Under an absolutist view, what is true and becomes truth stems from the axiomatization of knowledge into static theorems that further generate and validate the provability of statements; however, as it turns out there are mathematical statements whose provability cannot be established. For example Gödel's (1992) incompleteness theorem argues, from a set of axioms, there exist true sentences that are not provable (Musgrave, 1993) leading to a property of the absolute tradition as to never fully establish all mathematical truths. A view incompatible with one that mathematical knowledge is subject to individual and public revision and fully capable of enacting positive cultural and social change.

Conceptual Framework

In positioning a discussion of CML, it is important to consider existing perspectives on mathematical literacy needed to frame mathematical literacy in critical pedagogy. Traditionally mathematics teachers are expected to focus on procedural skills and mathematical competencies for technical understanding. Teachers give a lesson, give students time to work in class, and assign homework. Society valued such mathematics instruction as useful in scientific labor in society (Carss, 1986) and as increasing the de-skilling of professions:

> "shop assistants no longer need to calculate change, bank clerks need know nothing about banking, waiters and waitresses no longer work out bills, engineering is reduced to following blueprints: even computer programming, heralded only a short time ago as creating a need for a newly creative, mathematically-trained workforce, has become, in the hands of the large companies who employ programmers, largely a routinised and alienating activity. As technology invades all aspects of daily life, people actually need less- not more – mathematics." (Noss, 1994, p. 7)

With concern for improved means of mathematics education and initial conceptions of mathematical literacy, change is primarily concerned with introducing students to contemporary applications in a technological society (Howson, Keitel, & Kilpatrick, 1981).

This project draws on three ideas: equity, equality, and dominant mathematics. With respect to equity, consider the Equity Principle (National Council of Teachers of Mathematics, 2000), which puts forth that all students must have opportunities, support, and access to a challenging curriculum and teacher. In a typical mathematics classroom, this means diversity in resources is provided to ensure that "mathematics can and must be learned by *all* students" (National Council of Teachers of Mathematics, 2000, p. 13). If teachers wish to begin revising the pedagogy of traditional mathematics in addition to furthering equity in mathematics education, they need to consider ways to address barriers that constrain teaching from reforming the dominance of the individual and embrace an ethical perspective on diversity (Greer, Mukhopadhyay, Powell, & Nelson-Barber, 2009), in which respect for difference and cooperation are cherished.

Although equity deals with diversity, respect, and cooperation among students, equality in education has taken many shapes. Teachers often are exposed to and made aware of *in*equalities that focus on a lack of balance. Consider the current U.S. minimum wage, per hour, $7.25 (U.S. Department of Labor, 2007), compared to the hourly pay of Chesapeake Energy's CEO Aubrey McClendon (Wall Street Journal, 2009), $338; or consider discussion of unequal comparisons which *seem* to carry less bias: equal opportunity across gender and ethnicity, or how students enter classrooms with unequal life experiences, yet are supposed to leave with the same learning, the same outcomes. Working toward equality draws vivid depictions of hierarchical uncooperative social structures, which are homogenous and unequal.

In *A Nation at Risk* (Denning, 1983), there was a call, a push, a race against other industrialized countries to change education policy because "our once unchallenged preeminence in … science and technological innovation is being overtaken by competitors throughout the world” (p. 469). These “competitors” had overtaken our cultural narcissistic view of global “preeminence” in science, technology, engineering, and mathematics. The solution offered was a new and accelerated push for improved support for the teaching of mathematics and science, emphasizing mathematics that would enable the United States to maintain dominance at the top of the global ladder: pedagogy of dominant mathematics with a focus on mathematical productivity, procedural fluency, and utility; the kind of mathematics that cuts children off from the society, communities, and democracy they live in. *A Nation at Risk* dominated mathematics for the following 7 years and positioned the high school curriculum to be based on corporate and industrial interests. The United States needed more high school graduates with “necessary” mathematical understanding to fill society’s technocratic employment needs. The 1990s saw educator interest in becoming more student-centered and challenging the dominant mathematics view of efficiency and fluency, with emphasis on locating mathematics in realistic, contextualized problems (Gravemeijer, 1998; Treffers, 1991; Van den Heuvel-Panhuizen, 2001).

As concerns about more student-centered instruction emerged during the postindustrial era, educators shifted from needs of a technological society to the role mathematics plays in the world. This shift, recognized in the documents of mathematics professional organizations (Mathematics Council of the Alberta Teachers’ Association, 2005; National Center for Education Statistics, 1993; National Council of Teachers of Mathematics, 2000), included procedural skills and competencies, previously valued, and began encompassing mathematical processes (e.g., NCTM), enabling students to understand the role of mathematics in society as it relates to students’ personal and communal activities. This more process- and being-centered focus is termed the “social turn” in mathematics education (Lerman, 2000).

This social turn in mathematics education gave rise to rethinking the purpose of the mathematics educator from one of passively transmitting knowledge to students, to being able to contribute to positive social change by addressing social issues and political challenges in society, developing ways to overcome the effects of racism, sexism, and classism (Frankenstein & Powell, 1988). Researching mathematical pedagogy that employs contextualized problems is more student-centered than traditional pedagogy, and reflects the social and cultural awareness of student, teacher, and curricula. The *public educator* (Ernest, 1991) is the foundation for the conceptual framework for this study. The public-educator ideology is a social constructivist perspective providing a foundation for progressive mathematics didactics and instruction as it recognizes that "linguistic knowledge, conventions, and rules" (Ernest, 2010, p. 36) are the basis for mathematical knowledge. Second, interpersonal discourse is required as students negotiate and reframe knowledge from subjective/personal to objective/public. Third, such a process is understood to be a human activity in which mathematical knowledge is negotiated through debate, acceptance, and revision.

Figure 1 outlines this process. The initial stage is located where public ownership meets the private realm. At this stage, students participate and engage in shared classroom or peer discussion about achieving a commonly shared problem or task. It is here that students' perceptions merge with what is initially negotiated with others. As each student further negotiates personal meaning, the location of that meaning shifts from the private to the public realm. This occurs as personal meaning gets conventionalized in that the individual's meaning is debated among others, compared, and revised to better align with the meaning of others.

	Social location		
	Private realm		Social realm
Public	Individual's public use of sign to express personal meaning	Conventionalization →	Conventionalized and socially negotiated sign use (via critical response & acceptance)
Ownership	Publication ↑		Appropriation ↓
Private	Individual's development of personal meanings for sign and its use	Transformation ←	Individual's own unreflective response to imitative use of new sign utterance

Figure 1. Model of the construction of mathematical knowledge (adapted from Ernest, 2010)

Still within the social realm, as meaning moves from public to private ownership, the process of appropriation deals with how the student internalizes meaning just negotiated, whose ownership was public. Additionally, the production of knowledge transforms from the social to private realm. In the private realm, ownership of knowledge becomes private as the internalized prior meaning transitions and leads to new personal meaning. Through the process of publication, the newly generated meaning whose ownership is private now becomes public as it enters the community (classroom) and is finalized. At this last and new first stage, students form new meanings based on the publicized knowledge made available through publication.

Research Questions

The focus of this study is on understanding the core values and beliefs of CML educators who constitute a culture-sharing group. By this I mean that these educators largely follow a perspective that recognizes “mathematics knowledge is culture-bound, value-laden, interconnected and based on human activity and inquiry. … It is a critical epistemological perspective, which sees ethics, social, political and economic issues [as] strongly inter-related … [where] knowledge is seen to be the key to action and power, and not separated from reality” (Ernest, 1991, p. 197). Developing an understanding of the group’s themes will better suggest how this subculture works and provide analysis of classroom practices and beliefs.

My involvement was to observe 3 CML teachers during their professional working day over a period of 5 to 6 weeks. CML educators are participants in a subculture of critical teaching that engages in multilayered resistance to traditional pedagogy. For example, teachers educate for the purpose of social transformation and oppose dominant school mathematics (Gutiérrez, 2002), and may practice other forms of resistance at various levels. To investigate these practices, my study addressed two questions. The first question relates to the relationship between participants’ conceptualization of CML and their classroom practice. The second question addresses where the participants’ beliefs are situated. The questions follow:

1. How do CML high school teachers conceptualize their CML practices and how are those practices enacted in the classroom?
2. What is the nature of CML high school teachers’ instruction aligned with a CML philosophy?

Methods

The overall study approach followed an ethnographic perspective (Gobo, 2008; Madison, 2005; Spradley, 1980) in collecting data, observing, and understanding the meaning of participants’ interactions and behavior. This process occurred in stages that reflected changes in observation and coding. The first stage of observation was descriptive in nature, concerned with capturing the mathematics classroom and describing the teacher, the students, and the activities in which they engaged, after initial descriptive observations centered on locating the cultural patterns occurring in the mathematics classroom. For example, I observed the ways a teacher presents a problem, the mathematical activities, and the physical objects that were present. Questions and topics generated during the initial observations led to further observations that focused on investigating structures and relationships, such as the ways teachers used authentic data as part of a CML problem in their classroom instruction, or the students' specific responses to social or political mathematics problems. Observations at this stage are called focused observations.

This methodology implies changes in the relationship between time spent observing and the stages of observation. During initial observations, the majority of time was spent conducting descriptive observations. Starting with the 5th observation, attention shifted to focused observations. By the 10th observation, time spent conducting descriptive and focused observations was gradually reduced to allow time for selective observations. The relationships among observation time and observation type are detailed below in Figure 2.

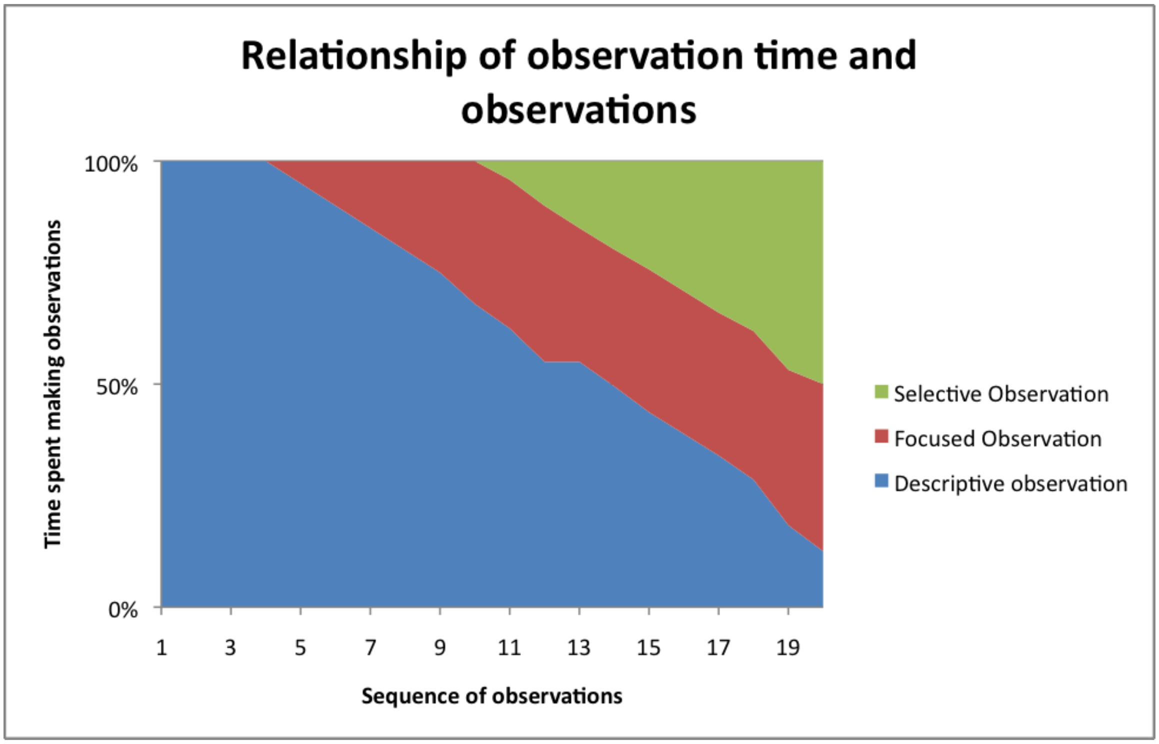


*Figure 2.*Relationship of observation time and observations

Sources of Data

Data were collected from five sources: field notes and a fieldwork journal, interviews with participants, classroom observations, examples of teacher work, and participant autobiography.

Field notes and journal. Field notes and a fieldwork journal were condensed, detailed, or analytic in nature. I used these as an opportunity to record ideas, experiences, breakthroughs, memoranda, and challenges as they occurred. For example, Appendix A exhibits the initial observation protocol and an example of a typical session.

Interviews. Two formal interviews were conducted with each participant. Follow-up and informal post-lesson interviews were also done in order to clarify and better understand any concerns during class time. All interviews were recorded and transcribed for analysis. The initial interview, outlined in Appendix B, was designed to probe the participant's educational ideology for mathematics through addressing such elements as how they describe their personal views of mathematical epistemology and views on goals of mathematics education in relation to modern culture and society. For example, when asked to describe a focus of instruction (Interview 1, Question 1a), Gwen mentions an emotive focus on mathematical knowledge:

> **Gwen:** I want [students] to have knowledge of mathematics. But I want them to have good feelings about mathematics. I want them to feel like it's not awful; it's not scary; it's not abstract. It's something that they can do, it can be learned in incremental pieces, and it's useful… I want a student to think to themselves, "When I'm doing math, I feel safe," or "I don't feel out of control or scared or incompetent." I want them to know that everybody who learns math well learns it from somebody.

The second interview, outlined in Appendix C, queried participants on their beliefs and notions regarding teaching and learning mathematics from a critical mathematical literacy perspective. For example, when asked about the nature of teaching mathematics, Owen responded:

> **Owen:** I find mathematics is something that we can utilize every day, and that's something that I preach quite a bit to the kids. That math becomes a piece that we can -- everything that we do involves math. So I tell the kids that when you're driving, just the idea that being able to stop your car in time and not go through the stop light -- that's math. I find that to me, in my mind, it started the idea of what are we teaching our kids? Does anybody realize how much math we really need just for even a job as mechanic? Usually, we think mechanic -- turn the wrenches and the job's done. Not so much anymore. I think that … as you look throughout time that for some reason, it's this innate idea of the math has always been there. In fact, people have always tried to get their mind around it. And that we advance it every day and that this thing that we call math.

Classroom observation. Classroom observations were conducted during each participant's regular school day for the entire six weeks. These observations progressed in stages from *descriptive* to *focused* to *selective*. The initial points of concern for *descriptive* observations are outlined in Appendix A. *Focused* observations centered on a theme or detail important from the *descriptive* observations. For example, below is a sample of my notes from a *focused* observation with Jack from November 22nd concerning how student interactions influence his instruction-

> One of the challenges with group work is to keep students on task. Clickers are a tool very suitable for engaging students and monitoring their progress. When they have answered a question, it often opens up further opportunities for class discussion, which can lead to smaller group discussions later.

Selective observations are the most narrow and, in particular, focus on a semantic relationship (see Table 1). These observations generally occurred during the last week of observation. With a primary focus on a semantic relationship, *selective* observations were used to further establish the given relationship across dimensions of contrast (see Table 1 below) or extend that relationship inwardly (see Highlighted Text from 12/6/2010 below).

Table 1

Selective Observations Dimensions of Contrast and Highlighted Text

<table>
<tr><th></th><th>Where is the focus?</th><th>Role of student?</th></tr>
<tr><td>Peer-principled ethics pedagogical tools</td><td>Self, peer</td><td>“Be responsible”; social amiability</td></tr>
<tr><td>Image-building ethics pedagogical tools</td><td>Self, peer</td><td>Receive/maintain T-S camaraderie</td></tr>
<tr><td>Métier-based ethics pedagogical tools</td><td>Work culture</td><td>Receive/learn culture</td></tr>
<tr><td colspan="3">Highlighted Text from 12/6/2010 Selective Observations with Jack.
This afternoon I am focusing on the qualities teachers need for building students' positive mathematical self-identity. Presently this involves “proper questioning” which looks like a series of probes and suggestions that Jack uses to get kids thinking about things more critically. Possible stages are (1) asking open ended questions, (2) creating space/wait time for students to consider answers to the questions, (3) considering with the class open ended (extension-type) questions, (4) opportunity for sharing.

An additional consideration for positive mathematical self-identity involves “socialization in the classroom” as experiences where the teacher is maintaining or</td></tr>
</table>

relaxing control of classroom structures such as organization, scheduling time for activities, etc. During this particular class (Grade 10 applied math) Jack makes such socialization individual by differentiating instruction to introduce and re-expose students to situations and skills needed in his community of learners.

Examples of participants' work. The examples of teacher work included lesson plans, professional or curricular materials, problems and examples, and materials used during teachers' daily professional life. An example of materials used during participant's daily professional life is exhibited in Appendix A-1. This exhibit is a rubric used by a participant to assess students' final presentation of a political or social issue across math, creativity and presentation, and information at the end of the semester of a grade 10 applied math class.

Participant biography. Participants completed autobiographies including how prior beliefs and experiences shaped their current work. Topics suggested included their current perspective on their teaching practices, how they arrived at such a point, how this relates to their own academic experiences, how they have or have not developed professional confidence, and how personal beliefs shape their participation in teaching culture.

Participants

Participants in this study included 3 full-time, veteran, secondary mathematics teachers. Selection for potential participants began with brief emails and/or phone calls to secondary educators that had expressed interest in engaging their students with social justice pedagogy. For example, I contacted mathematics educators on relevant email listservs or through conversation with concerned teacher educators. Initial efforts produced eight mathematics educators at five different districts/schools. After further conversation with the initial group, three participants were selected for full participation in this study. Final participants were not from the same school or district and have been secondary math teachers for 8 to 11 years. All participants were state/province certified professional educators in secondary mathematics. All participants had completed graduate level mathematics courses and attained graduate degrees in mathematics or teaching mathematics at the secondary level. One participant also had a graduate degree in chemistry. The participants self-identified with a CML-focused ideology in that their interest in CML extended beyond simply teaching a particular unit or lesson(s) from a CML perspective. Participants' schools were public schools in large metropolitan areas in districts supporting at least 10,000 students. School administration and math chairs of each school were supportive of participants—acknowledging the participant as a CML teacher—and encouraging the teacher to continue his/her instruction.

Jack Harkness[1], at Maria Montessori High School, taught secondary mathematics for nine years. His classes used a locally-developed curriculum focusing on processes of mathematics (such as those outlined for grades 9-12 in National Council of Teachers of Mathematics, 2000) further supported by technology and complementary lab exercises. Math courses he taught ranged from algebra to analysis (with trigonometry). Maria Montessori High School was the most diverse in terms of location and demographics. The school's neighborhood was mostly lower- to middle-class with an immediate population of 510,000. The school district managed 14 high schools with 44 elementary and middle schools. English was spoken by 44% of the population, with Punjabi being the next most common language (27%) and Gurjarati (8%) and Urdu (8%) representing the top four most common languages.

[1] Throughout this study, all participant and school names reflect pseudonyms.

Gwen Cooper, from Heinz von Foester High School, taught secondary mathematics for 11 years. Her classes used *College Preparatory Mathematics* (CPM) for algebra 2 and pre-calculus and Hughes-Hallett et al (2005) for calculus. A fifth class, Senior Inquiry, used a locally-developed curriculum specifically addressing sociopolitical issues for grade 12 students. Heinz von Foester School was officially closed in 2011 and all students and teachers were redistributed into local schools. The school's neighborhood was predominantly lower class with average family income (2004) of $30,000 and roughly 20% of families with income below the 2009 poverty line. The largest ethnicity group in the school's neighborhood was Caucasian (63%) followed by Hispanic/Latino (14%) and Asian (13%). Challenges to the school district and administration were planned during Fall 2010 (see Appendix G) and acted on in the Spring of 2011. Ultimately the district cited test score equity as reasons for closing the school.

Owen Harper, from Giambattista Vico Secondary School, wrote and taught with a locally-developed and locally-published curriculum—commissioned by the district school board and provincial Ministry of Education—specifically focusing on maturing students' learning skills and work habits (Ontario Ministry of Education, 2010) along with mathematics for citizenship. His classes were not split by content (e.g., one class as algebra, another as pre-calculus, etc.) but by grade (e.g., grade nine mathematics, grade nine applied mathematics, grade ten mathematics, etc.). Courses of all other mathematics teachers at the school and district were also designed this way.

Data Generation and Coding Methods

The study followed an ethnographic perspective (Gobo, 2008; Madison, 2005; Spradley, 1980), in which the scope of observation and coding changes and the end result was an understanding of the meaning of participants' cultural knowledge and beliefs used to interpret their experience and generate meaningful practices. The stages of observation were (a) descriptive observations, (b) focused observations, and (c) selective observations. The stages of data coding were (a) domain coding, (b) taxonomic coding, and (c) componential coding. This process is outlined in the figure below:

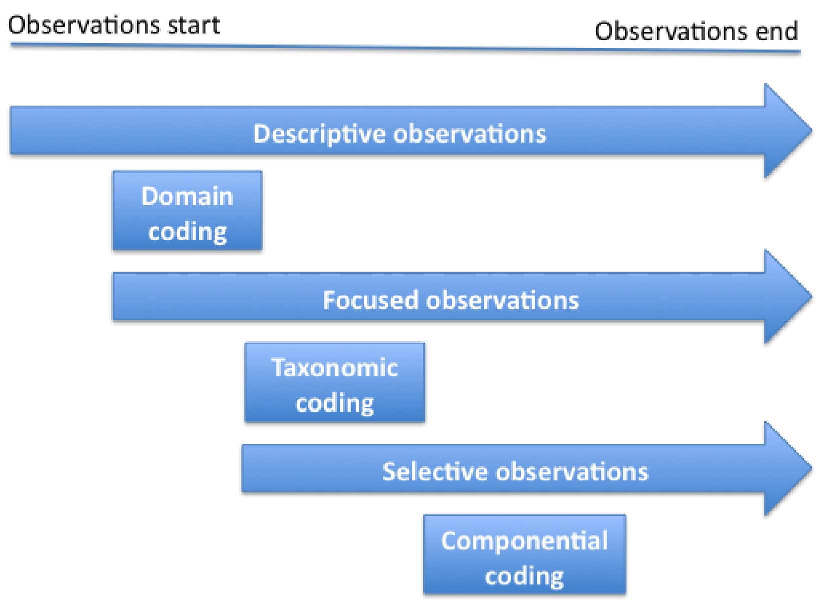


Figure 3. Semantic observation and coding relationship

Descriptive observations. During the first stage of observation, I was concerned with gathering descriptive observations of the physical space, the people involved, related acts people performed, objects that were present, activities people carried out, sequencing of events, goals people tried to accomplish, and emotions felt and expressed. These early stages of observation formed the basis of domains to be subsequently coded and analyzed. For example, during descriptive observations, I expected to observe the roles of the teacher in posing social or political mathematical problems and I expected to identify the kinds of mathematical knowledge teachers hoped to build with students in encouraging mathematical confidence to confront authorities.

Focused observations. Following descriptive observations, focused observations were conducted to study a smaller part of the cultural patterns and produce taxonomies that approximated and detailed participants' cultural knowledge by representing relationships among the parts of a culture. Here, observations were concerned with the ways components were structured and related. For example, prior domain coding had identified the means-end relationship of "motivating social change" as a domain Owen and Gwen frequently engaged in. The focused observation further expanded that domain to ask "What are all the ways to motivate social change in the math classroom?" Observation notes follow-

> *Across several participants (Owen & Gwen) there are different ways of motivating social change. So far I have observed two ways students have responded positively--*
>
> - *Soapboxing—In one instance, Owen questioned students about what it means to get an apology and how to know what is fair and what is not, and what about what are some employer assumptions about the students which could lead to racist situations. Through this deconstruction of a recent situation, he continues to highlight how the people with power will not be able to fix the situation if it is not brought to their attention.*
>
> - *Exhibiting— Gwen positions exhibiting in class conversation examining how data from the 2010 census might be different then 2000. The class considers how the white population is "so white" and how the Native American percentage of population is the smallest. Gwen asks students more about their city. She tries to get kids to think about how the numbers might be different. Reasons involve characteristics of cities in the West coast (students suggest less African American; lots of Asian in the neighboring city; age demographics might be different). Students are working with tables which include populations in homeless shelters and homicide rates. The task is to suggest two statements about what [students] observe. She starts conversation by noticing a difference in the homicide rates of 2004 (36% white male, 39% black men). The idea of what would be reasonable and equitable rates comes into conversation.*

Selective observations. In the last stage of observation, the scope progressed from broad to in-depth observations. At this final stage, observations were conducted to locate meaning in differences and similarities between teachers' specific practices. Selective observations drew on descriptive and focused observations and enabled the researcher to follow intuitive inquiry and test ideas in an attempt to understand the significance of related practices. For example, staying with the means-end relationship of "motivating social change", selective observations further expanded the ways to motivate social change through soapboxing as (part of) a classroom poster outlined below:

*Figure 4.*Exhibit of data demonstrating selective observation of "motivate social change"

Data coding procedures. The coding of data was conducted in three stages, following each of the descriptive, focused, and selective observations. At the first stage, data collected during descriptive observations were analyzed using domain coding to discover cultural patterns. Then data collected during focused observations were analyzed with taxonomic coding. Finally, information gathered from selective observations were analyzed using componential coding.

Data coding quality. To address concerns about trustworthiness and credibility of data, this study employed three separate processes in approaching the issue from an interpretivist criteriology (Hammersley, 2007; Seale, 1999, 2002) as a means to emphasize data quality. Two processes were the ongoing interactions between the researcher, each participant, and their generated data. The first, critical voice, as a means to disrupt dominant sociomathematical political views and prior knowledge; the second, heteroglossia, as a further means of transactional validity (Cho & Trent, 2006) emphasizing participant and researcher convergence upon analysis and interpretation of collected data; third, traditional means appropriate for qualitative judgment cases were employed to exhibit agreement between two coders; this included computing Cohen's kappa (1960) and the Perreault and Leigh (1989) measure as means of assessing the overall quality of data.

Critical voice. In an effort to acknowledge the work of others that often goes unrecognized, Madison (2005) suggested that the critical ethnographer employ positionality to disrupt neutral–status-quo politics and dominant knowledge. This, in turn makes accessible "the voices and experiences of subjects whose stories are otherwise restrained" (p. 5). In my study, the role of the participant observer has two purposes: "(1) to engage in activities appropriate to the situation and (2) to observe the activities, people, and physical aspects of the situation" (Spradley, 1980, p. 54). The role of the participant observer is to reflect on one's own actions, the behavior and norms of others, and what is happening in the social situation.

Based on these purposes, data gathering occurred as I reflected on actions and engaged in insider/outsider experiences. As an insider, I have experienced the challenges, successes, and frustration that teaching—particularly from a CML perspective—can provide. As an outsider, I viewed the classroom, students, teachers, and myself as subjects. The data collection occurred during both experiences. As such, the role of the researcher as participant observer involved moderate participation (Spradley, 1980) in which a balance was struck between insider and outsider, and between participation and observation.

Heteroglossia. Teaching from a CML perspective can be a sensitive issue for the teacher and the school. I believe my experience as both a classroom teacher and teacher supervisor has given me insight into this world. In keeping with the social and transformative nature of CML, analyses and interpretation are heteroglossic in nature (Dentith, 1995). As a means of working with participants, heteroglossia refers to the centripetal nature of discourse, in this case teacher discourse, "towards the unitary center" (Dentith, 1995, p. 35). This process may be thought of as similar to established methods of participant validation or member check. Heteroglossia is particularly useful in the case of CML, as participants help to improve on the accuracy and transferability of the study's findings. One of the goals of the project was to explicitly acknowledge the voice of this subculture of teachers. Heteroglossia will help to acknowledge and elaborate on the practices and strategies teachers use when engaging in the six domains of CML. It also allowed the teachers to contest and suggest ways for revising the understanding of the CML domains. This is of great significance, because heteroglossia enables teachers to work through their own personal beliefs and history to further establish these domains and arrive at a unified understanding. This centripetal process of arriving at a unified understanding is further validated by CML teachers' discussion of how their practices and discourse generate agreement and meaning making (Quine, 1960).

Overall quality. As a means in considering the overall quality of data, and to determine the amount of agreement in the study's qualitative judgments, Cohen's kappa (1960) was employed to observe the proportion of agreement between two coders. Using 20 randomly selected cases, the overall percent of agreement was found to be 90% and the coding across the cases resulted in κ = 0.88167. Further, using the overall percent of agreement as 0.9, ordered cases into 13 categories of coding, and across 20 random cases, the resulting Perreault and Leigh constant was found to be 0.94428. Generally, values between 0.8 and 1 suggest strong inter-rater reliability for both values and, in this two coder case of the Perreault and Leigh measure, a "better behaved" (Rust & Cooil, 1994, p. 3) value of .944 also suggesting strong agreement between coders.

Domain coding. During the first stage of analysis, I sifted through data to begin coding the cultural patterns (called domains) participants exhibited. The object of domain coding was to locate and identify the semantic relationships in the cultural domain. In general, nine semantic relationships existed in domains. These are presented in Table 2.

Table 2

Semantic Relationships

Form	Semantic relationship	Example
1. Strict inclusion	X is a kind of Y	Addition is a kind of mathematical operation.
2. Spatial	X is a part of Y, X is a place in Y	The overhead is a part of the classroom.
3. Cause–effect	X is a result of Y, X is a cause of Y	Student–teacher dialogue is a result of classroom discourse.
4. Rationale	X is a reason for doing Y	Gaining understanding is a reason for doing problems.
5. Location	X is a place for doing Y	The teacher's desk is a place for grading.
6. Function	X is used for Y	A calculator is used for graphing equations.
7. Means–end	X is a way to do Y	Letter writing is a way to do public critique.
8. Sequence	X is a step (stage) in Y	Checking work is a step in completing a problem.
9. Attribution	X is an attribute (characteristic) of Y	Posing social problems is an attribute of CML instruction.

Note: Adapted from Saldaña (2009), Spradley (1980), and McCurdy, Spradley, and Shandy (2005).

Taxonomic coding. Following domain coding, the second stage of coding—taxonomic coding—helped the researcher understand more deeply how contrasts in knowledge and behavior were systematized and organized. This was accomplished by expanding and revising a selected domain, then introducing subsets and additional terms based on similarities. For example, after domain coding classroom observation notes from Owen's classes during the first week of June, 2010, the strict inclusion semantic relationship "kinds of status tokens" was identified. Subsequent taxonomic coding then expanded this relationship in specifically coding data to determine the answer to the question "What are the different kinds of status tokens?" An example analysis is exhibited below in an interview with Jack Harkness-

Michael: Tell me about what you hear from different people at different schools who have tried to do similar things that you're doing, and what they've found successful or found challenging?	
Jack: It's definitely come across as a challenge. ***But it helps to have a starting place. So that a teacher that wants to implement a program like this, they go through a process of learning about how to teach social issues and social understanding of using math.***	**(What kind of status token is this?)** ***Pedagogical trophy***

Figure 5. Exhibiting a taxonomic coding session

Componential coding. The final stage of coding was used to organize and represent the meaning people have assigned to their knowledge and beliefs. This was accomplished by organizing and representing contrasts found in prior stages of observation and coding. During this final stage of coding, I was concerned with the meaning participants assigned to cultural categories. The example of a componential coding session, highlighted in Table 2, compares likenesses and contrasts found in prior stages of observation and coding (student- and teacher-centered) with two strict-inclusion semantic relationships ("kinds of spaces" and "kinds of status tokens").

Table 3

Sample Review of Componential Coding Session

	Student-centered		Teacher-centered			
	Involves student-centered social aspect?	Involves student-centered cultural aspect?	Involves teacher-centered social aspect?	Involves teacher-centered cultural aspect?	Involves pedagogical trophy?	Involves professional barriers?
Kinds of spaces	Yes	Potential	Yes	Yes	No	Yes
Kinds of status tokens	Yes	Yes	Yes	No	Yes	Potential

Domain coding identified "kinds of spaces" and "kinds of status tokens" as important relationships for participants. Through more observation specific to these relationships, and through taxonomic coding, student-centered and teacher-centered were identified as various kinds of "spaces" and "status tokens." Componential coding contrasts how such attributes are characteristic of each relationship.

Results

This section will present the results of analysis in two parts. The first part will present results specific to the first research question. That is, how participants conceptualized their practices and ways those were enacted in the classroom. These results are framed by ontology of the culture of CML teachers in which four kinds of spaces were exhibited by the data: ways to transmit culture, the Bardo, the CML narrative, and teacher structures. These results are outlined in the table below and are a reply to the first research question. The second part of the results section will examine the nature of participants' values and beliefs aligned with CML instruction. These results are a reply to research question two and are represented by four different aspects of each participant's mathematics education ideology. The four aspects examined are epistemology, the participant's interpretation of the role of mathematical knowledge, the participant's perception of the social-mathematical role of the student, and their value interpretation of the culture of CML teaching.

Table 4

Outline of the First Part of the Results Section

How do CML high school teachers conceptualize their CML practices and how are those practices enacted in the classroom?

Ontological space	Focus	Characteristics
Ways to transmit culture	Bottle-necking life opportunities	Actions school community members participate in that do not fully engage or challenge actors
	Maintaining the status quo	Messages and practices that guard dominant culture
	Image-building ethical pedagogical tools	Working with students so they become positive self-managers and begin to recognize their positive and negative behaviors
Bardo	The cultural Bardo	Ways in which teachers help students to realize understanding mathematics involves positive beliefs and emotions; experiencing mathematics through communities and personal experiences
	The social Bardo	Societal spaces, often stigmatized by dominant culture, yet important for students, teachers, and community members (e.g., prison, community poverty, the war in Iraq, the Navajo Nation)
The CML Narrative	Contexts	Everyday environments beginning the CML narrative, enabling data, discussion, and problems to be situated (e.g., data on tuberculosis deaths from 1990 to 2010, a school board meeting)

	Actions	Process of connecting to a Narrative's context in order to engage in thinking and learning about mathematical arguments (e.g., understanding a function's output in terms of income disparity)
	Results of actions	Close the CML Narrative based on observations and experiences (e.g., optimal placement of sleeping mats in a homeless shelter) or that has significant self-meaning (e.g., organizing students for a protest march)
Teacher structures	Didactical	Objects that represent the professional obligations, challenges, and roles participants faced
	Coping	Means and ways participants employed in dealing with stressors of teaching (e.g., classroom coffee station)

In this first part of the results section I will discuss findings from my study that suggest the need for mathematics educators to question the ontological perspective of current mathematics education thinking in relation to how mathematics education has historically developed against the backdrop of mathematics. Our thinking about this relationship has been greatly influenced by early Greek thought. For example, early examination of mathematical truths developed from the need for articulation and demonstration of those truths (Bishop, 1988) and that mathematics, as viewed today, can largely be framed on the foundation of the Greeks (D'Ambrosio, 2009). However, in the evolution of mathematical thinking, constructivist thought suggests learners may never arrive at some a priori body of mathematical knowledge (Jaworski, 1994) as, particularly in the CML classroom, mathematical knowledge is established socially (Ernest, 1998; Jaworski, 2001) through context, dialogue, and activity (Lave, 1993).

This foundation on Greek thought persisted, and still influences discussion of mathematical ontologies, through the modern era. However, recent thought in mathematics education occurring in recent decades argues that absolutist pedagogies—while important in framing an ontology of mathematics education—are not importantly influenced by concerns for culture (Bishop, 1988; Wax, 1993), language (Sethole, Goba, Adler, & Vithal, 2006; Wells & Arauz, 2006), and communities of practice (Lave, 1993; Lave, Murtaugh, & De La Rocha, 1984) in the secondary classroom. That is to say, as mathematics education researchers, the question of how one thinks about ontologies is crucial because it is from reorienting the ontological perspective that one can position CML in the field, as one shifts from perspectives that regard mathematics as absolute or formalist and is devoid of social awareness in the role of the individual in arriving at their own mathematical meaning.

From the data gathered during this study, several spaces that describe the general conceptions governing the plurality of what "can be applied to the claims and judgments of [the classroom] discourse" (Timmons, 2006, pp. 29–30) arose as characteristic of the ontological spaces teachers of CML and their students inhabit. These spaces govern, through discourse, navigation through the collection of objects, properties, facts, and conceptions—the members of the spaces. Further consider that the members of the spaces include the "customs, norms, attitudes, sentiments, and the aspirations that characterize" (Bernstein, 2010, pp. 73–74) human social interaction, relationships, and inquiry.

These ontological spaces comprise the foundation on which its members perceive reality. In mathematics education, these are the communities of practice constructed of cultural and social lived experiences by which mathematical meaning-making is not established a priori but instead via human social interaction, relationships, and inquiry. Learning mathematics is a process in which students continually revise their understanding in response to cooperative peer analysis and debate about mathematical relationships.

Ontological Space

Ways to transmit culture. This first space, *ways to transmit culture*, encompasses two practices that describe how culture is maintained, revised, and mediated. The first aspect, (a) *bottle-necking life experiences*, addresses the actions by which students participate in, and are challenged by, life opportunities. The second aspect (b) is *maintaining the status quo*. This aspect builds policy barriers and decisions onto messages the dominant culture wants to transmit and how the present states of these policies and decisions serve as status gates. The third aspect supported by data, image-building ethical pedagogical tools, briefly portrays how culture is revised and mediated through image-based ethical-pedagogical tools. These illustrate the processes involved in sharpening students' self-image in which both student and teachers engage to develop social skills.

Culture is maintained by bottle-necking life opportunities. Maintaining culture by bottle-necking life opportunities is about the actions in which students participate, and are challenged by, life opportunities. Consider the decisions students make to attend school even when it is not enjoyable. During a June assembly on school issues at Giambattista Vico Secondary School, several students unexpectedly entered the gymnasium late. This was brought to the attention of everyone present, as a different group of students chanted "Skip; skippy!" The negative badinage from students notwithstanding, the late students recognized their poor judgment and did not give up, returning to school. During another assembly a few weeks later, still at Giambattista Vico Secondary School, an administrator mentioned that graffiti belonging to the MS13 gang was found; the administrator continued, emphasizing the school's anti-gang policy and a $100 reward for information. With students in such an environment, no wonder they may not feel challenged or engaged to participate in classroom activities in productive ways. Student badinage, poor judgment, gang violence—these are barriers that delay and ground students to life opportunities. When students chanted “Skip. Skippy!” there was no response from adults; chastising students during an assembly as a response to graffiti tagging dictates rather than yielding to suggestions from those affected on how to handle the issue; low student morale in the classroom benefits no one and keeps students from seeing that teachers do care and society does offer thoughtful ways to deal with conflict.

Culture is maintained by maintaining the status quo. This second aspect of ways to transmit culture describes policy barriers and decisions onto messages the dominant culture wants to transmit and how the present states of these policies and decisions serve as status gates. These jointly guard the dominant culture from minority use. In this respect, as a way to transmit culture, dominant culture is not necessarily transmitted but *maintained* by efforts to continue the hegemony of the status quo. These policy barriers, decisions, and messages from the dominant culture jointly guard the dominant culture from a minority one. In this respect, as a way to transmit culture, dominant culture is not necessarily transmitted but used to *maintain* the hegemony of the status quo. In considering the meaning of a status gate I am discussing traditions, such as the societal belief “that well-educated people think they don’t need to know much math, but that having some knowledge of it is used to keep people from being more educated” (Follow-up interview with Gwen Cooper on October 30, 2010). This emerged across coded data from observations in Owen’s and Gwen’s classrooms and referenced largely how educators shape students through an enculturation process in which they (students) begin to believe in, belong to, participate in, the specific practices of a (predominantly white) culture. This "playing/being white," as evidenced during a school assembly, develops as students are awarded special privileges for achievement or made to feel as part of a "subculture" depending on which grade or classroom they belong to (June 14, 2010 observations with Owen Harper and October 11, 2010 with Gwen Cooper).

Culture is revised and maintained through image-building ethical pedagogical tools. Image-building ethical pedagogical tools' primary role is to provide a medium through which culture is revised and mediated. This examines the process of image-building; these may be thought of as a method to help students with self-image building, specific to a CML classroom. This involves means by which "teachers try to work with students so they may become positive self-managers and begin to recognize for themselves their own positive/negative behaviors" (Gwen Cooper, October 14, 2010). A part of the image-building process looks at metaphorical concepts in the classroom. For example, a high level of student respect, social interactions, and positive peer-group work flows smoothly and students know what is expected of them in established roles.

Classroom organization and reorganization are easily maintained so that students quickly arrange into groups to be better able to dialogue with each other. Consider the following example, from Owen's class (June 29, 2010), which started as Table 2 was outlined on the class whiteboard.

Table 5

Class Rules

Poverty-hidden rules	Middle- & upper-class-hidden rules
I know how to get someone out of jail.	I know how to hire a private lawyer to handle criminal or civil matters.
I know how to physically fight and defend myself.	I know how to reserve a table at a fine restaurant.
I know how to entertain a group of friends with my personality and stories.	I know how to set and decorate a table with centerpieces, place mats, and napkins.
I know how to get put on public assistance.	I know how to evaluate and purchase appropriate medical, life, disability, auto, or other kinds of insurance.

The point here was not to make students feel badly about themselves but to focus on what it takes to be in each of the groups. Owen's classroom discourse considered whether and how students will be able to overcome their membership in the poverty group and begin participating in the middle class by acknowledging such hidden rules. And, while the lesson was not overtly mathematical in nature, it did involve image-building as an ethical pedagogical tool, as the class realized that people in poverty are spending money to live and on survival items while the middle- and upper-class group tended to spend on luxury items or things that were not needed for survival. Through this process the teacher mediated students' understanding of social structures present in society and worked with them to revise their cultural understanding as they considered how one begins to understand and question the difference between the two groups' hidden rules. This development of qualitative assessment by the teacher draws on students' qualitative reasoning abilities and capacity for inquiry.

Qualitative assessment plays an important part in the image-building process of the students in a CML classroom. Qualitative assessment may take the form of presentations, papers, written self-assessment, or rubrics. Through such assessment, students may have to do the same mathematical practices over and over, the teacher may mark and verbally assess that practice, through feedback encouraging students to modify their practice until they are more adept. The goal is that both teacher and student will know that the student's understanding is where it needs to be. The idea of image-building tools as metaphorical is that the form of pedagogical tools deals more with social organization and interactions to contribute toward the image-building process of student–teacher and student–student interactions and camaraderie. For example, in addressing the notion of competition for college scholarships, Owen suggests, during a post-observation follow-up, for educators to support their students negotiation and understand of their own image as "what they don't know is how do you go about it, and making them adhere to a middleclass, white, fully fed, fully healthy person's world isn't going to match that underprivileged, underrepresented kids world. Those are two separate and distinct places. You can't expect this group to just -- to plop them in here and say, oh, you're going to be fine."

The Bardo. The Bardo is a state in which existence is located between two extremes. In the context of this study, it becomes a continuum, or timeline, of individual experience with CML. It is the gap that exists between being completely, critically mathematically illiterate, and fully literate. As a space that individuals pass through, it is charged with the adversity and challenges that individuals confront as they progress toward the complete experience of becoming fully developed in the understanding necessary for CML. In the state of Bardo, both in the cognitive and the psychological sense, one experiences their own false beliefs, fears, and negative emotions that prevent them from being fully developed, mathematically literate individuals.

In describing her own interpretation of the Bardo as an ontological space, Gwen comments that

> It's the positive mathematical development of students as a process students become fully aware of how to become mathematically successful [and includes] … developing students' sense of mathematical confidence by undoing the damage from previous math teachers. This is something [I] get better at year after year and can thus work better towards helping students achieve mathematical improved knowledge. One of the goals … is also getting students to take more responsibility, mathematically and otherwise, for themselves and to move forward in personal growth. Similar to the process of a parent helping their toddler learn to walk: first there is the parent holding the child as the child walks, then gradually the parent lets the child go as the child becomes more aware, recognizes more, that she can be self-independent. … The idea [is] that it's a process of the student unlearning bad math behaviors, of the student becoming more mathematically independent more literate. That for the teacher it is a process of being patient with the process of getting to know the student, developing a kind of "radar" by which I can sense if kids are there.
> (October 22nd and 29th, 2010)

Another document, from a professional development Jack attended, frames the space through which students develop critical mathematical literacy countering illiteracy as a meaningful barrier to be overcome:

> When a student is in this course, they should not ever have to ask, "When am I going to use this?" Every activity incorporates meaningful, engaging and enjoyable uses of mathematics that can be applied to life outside of school. That is not to say that it is an easy course to go through. This is, in essence, a course in critical thinking through mathematics. One of the major challenges for students is to struggle long enough to develop their critical thinking skills sufficiently to be successful. It can be quite frustrating to think that one understands all that there is in a situation, only to find that there is an entirely different point-of-view that went unnoticed. With time and patience that barrier can be overcome in all students. …
>
> Success is possible for all students. This is a critical aspect to the philosophy of all courses, but is especially important here as many students in this course will not have had success in mathematics before. There are numerous opportunities to demonstrate mathematical skills in a variety of ways and it is important for students to focus on their

> learning and not their marks. …When a student has successfully completed this course they will have enjoyed mathematics, possibly for the first time in years, they will have learned about mathematics as a tool for critical literacy outside of school.

These data frame the idea of a Bardo as beginning when a student first experiences a mediocre math teacher; that at that time, the student's sense of mathematical wonder gets reduced to minimal and it is then that students develop and form their resistance to positive mathematics instruction.

While the student must make his or her own progress through the Bardo, his or her teacher can ease the student's resistance by undoing the negative mathematical self-image produced, although not exclusive to, previous negative math experiences by supporting and guiding students. Consider how the student's negative mathematical self-image is supported by self-delusion ("I'm not good at math"; "I'll never understand"; "I quit now. It's too difficult.") and the need to defend one's self from our own self-fears which twist and turn students' mathematical experiences into acceptability blocking students from truly knowing a positive mathematical self-image and experience for what it is, instead of what they need/want it to be.

In speaking of the journey that students and teachers map toward reversing negative self-image in terms of mathematics, Jack comments (December, 2010) as follows:

> **"**Everyone has the capability to understand the importance of mathematics on their own terms. And that could mean any number of different things. There's certainly an environment aspect of if they have parents that are supportive of the value of math, or if they have parents that are math-phobic, then that -- how could that not have an influence. I am quite convinced that if you have supportive parents, you have a better opportunity to develop those math skills than if you don't [and those math] math skills can be developed in the right sort of context… One context [is challenging that phobia] and [I tell my students to] talk to your parents and list all the things they do that are related to math in a day. If that happened right off the bat, if a child in junior kindergarten comes home and says okay, mom or dad, listen, I've got this assignment you've got to do and help me out here. What are all the math things you do in a day? And if they start thinking about, if the

> parents think about that for 12 or 13 years, then eventually, it will sink in that they do a lot of math."

The Bardo emphasizes the nature of non-duality; that is to say, we overcome negative self-image by not experiencing it; we come to understand mathematics by experiences not realized as mathematics. Everyday our students are interacting with mathematics that is all around them. It's part of the world and so many of an individual's experiences with math are not even realized as math, particularly as with children. Consider the practice of counting: when one child complains that someone got *more* than they did; "They got two and I only have one." In this, math is experienced but it is not yet as separate as it will inevitably become as when mathematics becomes a subject in school.

Often, however, non-duality will not be recognized or suggested and, when someone is only trained in *teaching* mathematics, the structure and content of that teacher's instruction will further separate math from the real world where it exists around us and impress upon students opportunities to develop negative associations with math and to have them develop an identity that has them regarding math as something they "do not like" or "are not good at". The relation to the Bardo is that in recognizing the negative mathematical self-image as projections of the consequences of prior experience, which have been negatively reinforced through the actions of an inept teacher, in overcoming these past delusions math can be experienced in a *new* way in that it is not *bad*, *separate*, or an enemy--it is something that exists and can be understood. That it is something to be curious about and study and that, guided by a good instructor, one can learn, appreciate, and approach with an openness that is not marred by delusion/fear. It is then part of the role of the CML teacher to help students re-connect with math and to approach teaching in a way that helps students understand the subject matter as it relates to, and is experienced by them.

That the teacher *can* help students find new meaning and a positive mathematical self-image so that learning can take place positively and students recognize their own style of learning to more clearly see and direct their learning in more authentic ways. Connecting with that more direct and authentic means of learning was revealed through my data as the existence of two Bardo realms: cultural and social.

The cultural Bardo. The metaphor of the cultural Bardo begins with communities. These communities start with

> a very close-knit group of people when [a student is] growing up. That means a small neighborhood, and your family and friends. However, when you begin college, that community widens; as you meet more people, and the more educated you become, the bigger the community. As that community increases, the smaller the center community—who you are—decreases. (Owen Harper, June 8 Interview, 2010)

And on the basis of initial cultural Bardo formation: “It is shaped during [the student’s] childhood and familial environment. If Mom and Dad were not active in advancing their [child’s] education, learning of mathematics is going to get set aside” (Owen Harper, June 17 Interview, 2010). In interpreting a student’s status in the cultural Bardo, Owen said,

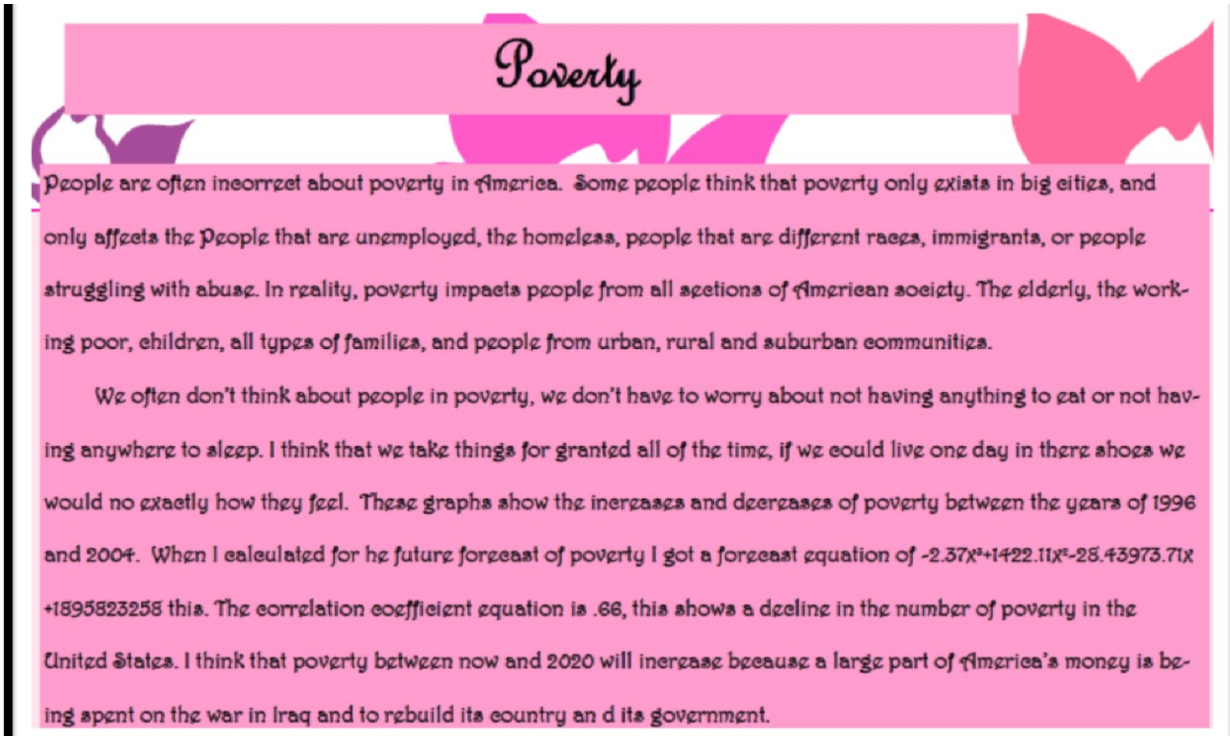
> You can move from lower class to upper class or from the underprivileged to the privileged…then can go back and help others to do the same. Through education and hard work, you can move into this privileged class but doing community work and helping others enables you to not lose your roots, and not lose who you are. (June 8 Interview, 2010)

In understanding the cultural Bardo, recall that the individual’s projections—the false beliefs, fears, and negative emotions—began with the initial negative experience developed by the inadequate teacher. However, the journey through the Bardo presents opportunities for individuals to discard these past delusions and to approach and experience mathematics in a positive manner. With a gifted teacher—be it in mathematics, driving, or cooking—students’ curiosity can be stimulated so that these negative feelings instead inspire confidence. In CML, a gifted teacher helps students appreciate and reconnect with mathematics in ways that help them understand how mathematics can be positively experienced in more personal and authentic ways.

The social Bardo. Although the cultural Bardo embodies students' beliefs and emotions, which are established and reinforced through negative interaction, the social Bardo consists of metaphor, physical locations, and processes that influence student passage. The social Bardo locates resistance to society, possibly by choice, and transformation through teacher guidance. In this social context, choosing to resist is categorized, for example, as being the first in your family to attend high school, or even making it past the awkward, adolescent educational stage.

I asked about Owen's school. "It's 100% Title I, with about 40% Native American, 40% Hispanic, and the rest a mix of African-American, Caucasian, and Asian, the smallest ethnic group." We discussed some student-produced materials that address poverty, border crossings, and biology. After reading the part on poverty, I wondered what it felt like to be the first in your family to attend high school and mentioned this to Owen. "Several of my students had previously dropped out of eighth grade but are now in my class. Even many of their parents did not finish high school." He added, "these first-generation students are the future. [Around here] it is [remarkable] to be the first person in your family to attend high school" (June 10 Observation Fieldnotes).

On the matter of non-choice resistance, the categories in the social Bardo are poverty, prison, "the Rez" (Native American reservations), Title I, and homelessness. Poverty is situated across several data points:



Poverty

People are often incorrect about poverty in America. Some people think that poverty only exists in big cities, and only affects the People that are unemployed, the homeless, people that are different races, immigrants, or people struggling with abuse. In reality, poverty impacts people from all sections of American society. The elderly, the working poor, children, all types of families, and people from urban, rural and suburban communities.

We often don't think about people in poverty, we don't have to worry about not having anything to eat or not having anywhere to sleep. I think that we take things for granted all of the time, if we could live one day in there shoes we would no exactly how they feel. These graphs show the increases and decreases of poverty between the years of 1996 and 2004. When I calculated for he future forecast of poverty I got a forecast equation of $-2.37x^3+1422.11x^2-28.43973.71x+1895823258$ this. The correlation coefficient equation is .66, this shows a decline in the number of poverty in the United States. I think that poverty between now and 2020 will increase because a large part of America's money is being spent on the war in Iraq and to rebuild its country an d its government.

Figure 6. Poverty data sample (adapted from Curriculum Development Materials, Maria Montessori High School, June 9, 2010)

The text says, "poverty only exists in big cities," but it also affects Native Americans on reservations. In collected data regarding social justice projects at the Mathematics Department at Heinz von Foester High School:

> In third-world countries, poverty is a big part of life. A strong woman referred to her experience, saying, "Poverty is like the blood through my veins." Poverty has been there since before she was born, and she continues to struggle to keep her family fed, safe, and alive. She said she would starve to keep her grandchildren alive.
>
> Poverty is watching your mother, father, and relatives die in pain. Starvation and disease often lead to death. Grandfather and grandmother crying out, wishing for death because they can no longer live in poverty. Poverty is watching your grandchildren die in your arms and you can't do anything but cry. (Observation Fieldnotes, December 2, 2010)

Prison and the Rez are locations of existence in the social Bardo. The common thread among these categories is that they encompass resistance-not-by-choice. By this, I mean that inhabitants often have not chosen to permanently reside in these locations. The Bardo description of prison creates challenges for student and teacher. For example, students who have a family member in prison have been categorized by society as unwanted or undesirable; similarly, inhabitants of Native American reservations have historically been free to participate in their respective societies and traditions, whereas, contemporary society has structured borders around these cultures and removed indigenous traditions.

One way in which the Rez category presents itself is when speaking of professional pedagogical challenges.

> The issue of "the Rez" comes up during a department meeting. Several mathematics teachers pose questions [critical of new policies] about the Bureau of Indian Affairs. Teachers discuss the issue of [getting speakers of Navajo as teachers to address] bilingual subject-content acquisition from students who transfer students from "the Rez." (Fieldnotes, June 25, 2010)

> **Michael:** Could you speak to some of the social justice topics involving indigenous peoples?
>
> **Owen:** [One of] my students did a mathematics project on native peoples'… workers' rights and pay as well as conditions. … We got onto the topic of the reservation. We were … talking with students from the [Navajo] Nation about some new programs there … [such as] the possibility of free solar and high-speed Internet. The students pointed out: Why would I need high-speed Internet if I don't have a computer? And, what good does a computer or TV do when we still have no electricity?" (Owen Harper, June 25 Interview, 2010)

The social Bardo theme of homelessness occurs in numerous data points across several participants. In one instance, the theme was used as a setting for a geometry problem.

> The teacher begins today's lesson, which is about exponential growth. Gwen writes $y = ab^x$ on the OHP, and asks the students which variables are the start- and multiplier values. The students reply. Gwen continues by asking what they know about tuberculosis [TB].
>
> "It's a disease that affects the lungs!" shouted one student. Gwen Cooper asked a student to read the text and students gasped when they heard that one-third of the world's population is affected by TB.
> "It's not treatable," a few students called out, and there was discussion about access to health care in different countries. Continuing, the teacher explained how many homeless shelters in these countries are run by nongovernmental organizations and people have to

> sleep on mats. Often, these shelters do not even meet UN public-health standards for refugee camps (the text of this problem is in Appendix D). The teacher then asked students how many sleeping mats, which are 2 m x 0.75 m, will fit on the shelter floor. Students calculated the time, in days, of the number of people (hypothetically) infected with TB.
>
> In plotting the values, students use their results to interpret the graph as approximately exponential. The teacher outlines several students' solutions (diagrammed layouts of mats on the floor) while the rest calculate the total available floor space. They then compare the calculated rates of people per square meter in the shelter to UN standards for refugee camps (4.55 m per person). (Classroom Observation Fieldnotes, October 6, 2010)

Finally, the last social Bardo space is that of resistance-by-choice: transformation. During a professional-development session with his department, Jack Harkness indicated that this theme came up. In speaking to the positive effects expected of his 18 teachers, the following was discussed.

> Successful implementation of policy depends on the professional judgment of educators at all levels, their ability to work together, and to build trust among parents and students. It also depends on the continuing efforts of strong professional-learning communities to share their understanding of [content and] policy in developing effective implementation practices. [Student success] depends on creative and judicious differentiation in instruction to meet the needs of all, and on committed guidance from … leaders, who coordinate, support, and instruct … teachers. (Classroom Observation, Giambattista Vico Secondary School, November 19, 2010)

Gwen Cooper recalled a personal conversation with students. "Often, when I speak with my students about my experience at [a previous school] and how I felt disconnected and that there wasn't a sense that my students needed to know mathematics." During a parent–teacher conference, one community member said,

> Well, those poor kids. Don't they know that they are going to need [school mathematics]? Don't they know that to be successful, to move out of [the low SECS of the neighborhood, school, and students] and to succeed that you need to know mathematics? Poor them; they do not understand that. (Gwen Cooper, October 14 Interview, 2010)

In response, Gwen Cooper replied:

> Maybe my willingness to learn mathematics is that I am coming from that situation and want to improve, work harder, and understand more. … It reminds me that [mathematics does hold] high status and it's socially acceptable to know about it. In my college years there wasn't a ton of professional women, and most of them who went into high-powered professions were choosing medicine or law—for financial and status reasons. I try to encourage all my students to become agents of change, whether it's as an engineer, mechanic, an astrophysicist, or something else. (October 19 Interview, 2010)

For Gwen Cooper, it was important to address equity issues with her students. The district had decided to close her school, which had not met targeted guidelines; however it was later revealed that other issues were the problem. The structure of the school was unique: some mathematics teachers got to know their students and families well. Gwen Cooper also personally felt that it was imperative to stand up for her beliefs. As an activist/role model, Gwen Cooper taught the students important issues beyond mathematics.

The critical mathematical literacy narrative. The *Critical Mathematical Literacy Narrative* is the third *kind of space* in which existence was mathematized through the data. The narrative consisted of three processes: contexts, action, and results of actions. Contexts position and situate the narrative in everyday social, cultural, historical, or factual environments. Action, in the context of the Critical Mathematical Literacy Narrative, is where teachers position students to engage in inquiry-based analysis of the context. This is the heart of the narrative, and where most mathematical understanding occurs. Results of Actions is a summarizing structure in which teachers regroup students to consolidate students' mathematical understanding of what they are analyzing.

Contexts. We can think of the context as the location situated by everyday environments that students examine through the further parts of the narrative. Data suggest that contexts may be divided into four different environments:

- Factual—based in actual or released public data from government, nongovernment, or private institutions (e.g., tuberculosis death rates released by the Wisconsin Department of Health Services);
- Activism and counterculture—as resistance to the status quo and hegemonic cultural identities;
- Historical—involving conceptual ideas or beliefs, events, or sociohistorical institutions (e.g., the American Civil War);
- Popular culture—as represented in media, music, television, film, or newspapers.

Factual contexts. One example of a factual context demonstrates a class-created project, shown in Figure 7, which investigates the proportion of capital convictions in death-penalty states where the prosecuting district attorney was Caucasian, African-American, or Hispanic.

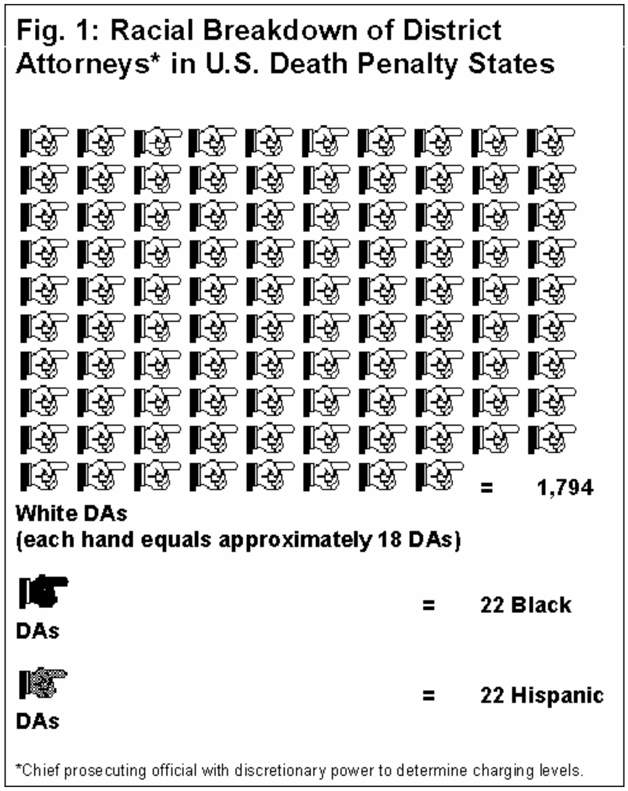


Figure 7. Ethnicity of district attorneys

Another example of a factual context involves a lesson conducted on October 25, 2010, in which Gwen Cooper had her precalculus students use data from the U.S. Bureau of Labor Statistics to compare rates of change across gender groups and years (see Figure 8).

Derivatives of Minimum Wage

Estimate the derivatives at the following times:

	Men & Women	Men	Women
1980			
1981-82			
1990-91			
1996			
2000			
2003			

What do you notice about the comparative rates of change among different groups? Describe at least three things you notice.

What do you notice about the rates of change for different years within a single group? Describe at least three things you notice.

Figure 8. Comparing rates of change across year and gender

Sample graphs shown in Figure 9 were displayed to promote further discussion:

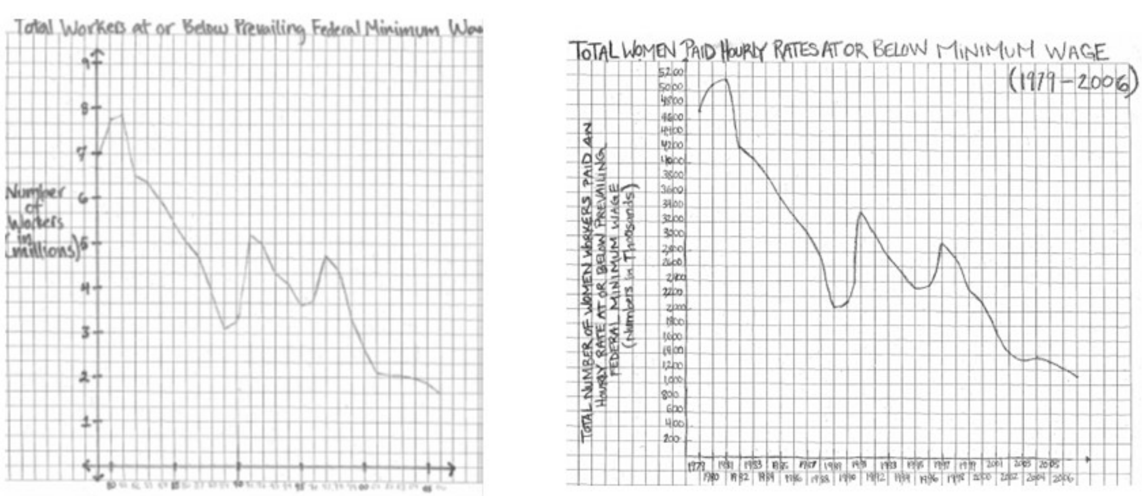


Figure 9. Sample graphs exhibiting gender income equity across years

Activism and counterculture contexts. Activism and counterculture contexts are concerned with everyday events and environments in which CML can be used. For example, Gwen Cooper gave a speech at her local board of education in response to a town meeting that announced the closure of several district high schools, including hers. The board claimed all students would be treated equally in decisions about their replacement school. Gwen Cooper, along with other mathematics teachers, was very concerned. The entire speech is documented in Appendix G. Another example of a counterculture context can be seen in Jack's comments on society's acceptance of mathematics phobia:

> One of the things that's frustrating is that people don't see how much they actually use math [on a daily basis]. If they did, then they would value it more. [To counter this] one of the things I've occasionally done with 10th and 11th grade students is [have them] talk to their parents and list all the things they do that are related to math in a day. Just think if that happened in junior kindergarten. A child comes home and says okay, Mom or Dad, I've got this assignment. I'm supposed to think of all the math things you do in a day. If parents and kids did this for about 12 or 13 years, it would sink in that they do a lot of math. (December 16 Interview, 2010)

Historical contexts. As a context in the Critical Mathematical Literacy Narrative, historical situations and environments are represented by conceptual ideas, beliefs, or events that may be sociocultural or sociohistorical in nature. There are several instances when participants used such conceptual material for lessons. First, Owen cited contexts of a "representative republic;" "free market economy;" and history books as "written by the winners." He alluded to provocative themes surrounding social acceptance of the Patriot Act in a historical context (Observation and Follow-up Interview, June 7, 2010). Above are examples of Gwen Cooper's use of historical data from the Bureau of Labor Statistics as a lesson for her precalculus students.

Another example of an historical context can be seen in Jack's use of an article claiming that the odds of life on a newfound Earth-sized planet are 100%. His ultimate goal was to encourage students to approach mathematical thinking not by explicitly seeking correct answers, but to "really think mathematically about what the article means and what the author is trying to say and, how the article can be supported with mathematics" (Follow-up Interview, December 2, 2010)

Popular culture contexts. Popular-culture contexts are represented by environments that students may find in everyday, lived society; specifically, they may stem from music, television, films, book, or other media. Early in observational data, popular-culture contexts consisted of situations such as "going green," and class discussions of economic- and societal value or the futility of wind farms versus coal-fired power plants (Classroom Observations, June 8 & 15, 2010, Maria Montessori High School).

Popular-culture contexts in Gwen Cooper's mathematics classes encouraged students to reflect on contemporary affairs from a mathematical perspective. During one interview, Gwen Cooper recalled a lesson of particular interest.

> A couple years ago, my class was examining exponential growth and decay. I had been reading something about Zimbabwe. … This lead to a discussion about what a currency is [and the role] of inflation, which is a tricky thing for students to understand, and the exponentials are a challenging mathematics topic. But it was really interesting to look at those numbers … thinking of the implications on [bordering] nations when the currency was skyrocketing with inflation. I thought, oh, I can use that in my classes. (Follow-up Interview, October 30, 2010)

Here, the current-affairs aspect of the popular-culture context was not important, but by using this relevant topic, Cooper was able, through a means appropriate to her class, help her students understand exponential growth and decay.

Actions. As represented in the data, there are six different kinds of Action that take place in the Critical Mathematical-Literacy Narrative. This second part of the narrative is where teachers work with students through a process of connecting their activities to the narrative's context. Ideally, the teacher can monitor pedagogy to help students arrive at mathematical understanding through several possible solution paths. This is the core part of the narrative; a point at which the mathematical process is rooted in several styles of thinking and learning and when classroom learning should take on characteristics of dialogue in which meaning-making can be constructed, not by blindly accepting the meaning given by peers, the teacher, or even the curriculum, but through social negotiation (Burton, 2004).

Creating awareness. Creating Awareness situates mathematical thinking and classroom learning in objects with which there may not exist much cultural or social understanding of how or why something is happening, or perhaps why it is an important topic for consideration. The first instances of this in the data stemmed from Owen's discussion of paper companies that prevent the planting of hemp: the class followed [Owen's remark, "How do we know the statistics [on hemp plantation prevention] are correct?" They questioned why it is illegal and what process would be needed for resistance to paper companies' lobbying to prevent permit shifts in existing laws (Classroom Observation, June 7, 2010). The mathematics taught that day in this context were not traditional mathematics skills but more about acculturating students to critical thinking and encouraging argumentation skills, as demonstrated by their comments on how to decide if "the statistics are right" and what skills would be necessary to provoke change.

Gwen's use of creating awareness took place over a series of 3 days for her algebra students. She started the unit with a graph of CEO pay, shown in Figure 10.

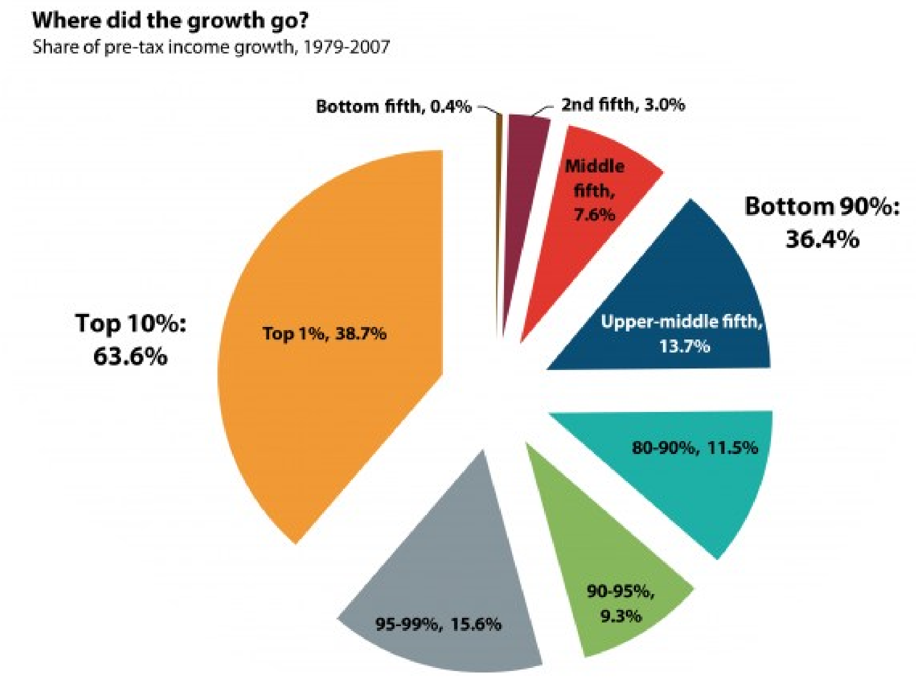


Figure 10. CEO pay

The lesson continued and became quite interesting as discussion moved to address Table 6, below.

Table 6

Inequality Index

- Percentage of U.S. total income in 1976 that went to the top 1% of American households: 8.9; percentage in 2007: 23.5.
- Only other year since 1913 that the top 1% share was that high: 1928.
- Combined net worth of the Forbes 400 wealthiest Americans in 2007: $1.5 trillion.
- Combined net worth of the poorest 50% of American households: $1.6 trillion.
- U.S. minimum wage, per hour: $7.25.
- Hourly pay of Chesapeake Energy CEO Aubrey McClendon, for an 80-hour week: $27,034.74.
- Average hourly wage in 1972, adjusted for inflation: $20.06; in 2008: $18.52.
- Median household income in 2008 was $50,303, according to Census data. Half of American households had income greater than this figure while half had less income.

Note. Adapted from "Inequality by the Numbers," by Institute for Policy Studies (2009).

The lessons examined salary and wage data forming three plausible conjectures per group (in the precalculus class), revising their conjectures as class discussion evolved, and finally reassessing their working conjectures to address issues such as "There are less White people working for minimum wage than others;" and, "More Whites are working;" and, more "men work full-time and more women work part-time" (Classroom Observations, Heinz von Foester High School, October 11, 2010). In this action, the space of awareness was created; however, as a researcher I felt it would be hard to position a lesson with a more advanced class such as calculus or IB HL 1/2. The real value in this Action is that the teacher framed reasons for decisions made by society, culture, and elected representatives through mathematics instruction.

Mathematically interpreting data. Mathematically interpreting data is the action in which classroom discourse takes shape in a space through which mathematical interpretation may solidify relations between the mathematical understanding and ordinary language. In other words, mathematical understanding in the classroom brought about through student interaction, discussion, and struggling with the environment in the context deepens students' comprehension into meaning that they understand in ordinary language. The action of mathematically interpreting data occurred in two differing modes of discourse that emphasize the students' understanding.

As recognized by Kinneavy (1971), these modes of discourse are expressive and persuasive discourse. Expressive discourse in the mathematics classroom may include activity-composition books, mathematical binders or journals. Expressive discourse may be typical classroom conversation that comes primarily from students questioning a mathematics problem, or statements such as, "I just don't get it," and prior experience with school mathematics (as organized by a mathematics course grades/transcripts, comments from prior teachers and students, and statements from parents about their child's mathematics ability). This example of expressive discourse draws on statements from students during Owen's algebra class' use of national crime data provided students with the following verbal example:

- If 18 year-olds account for 92,387 violent crimes, and the total number of violent crimes was 327,619 in the U.S., what percentage did the 18 year-olds commit?
- According to the data, juveniles are responsible for one-quarter of all crime. How much crime is committed by adults?

> Each group was given a different section of data about violent crimes, offenders, SECS, gender, etc. Students work with their groups to summarize the content of each of their sections. They write in their math binders and one group member tries to consolidate the others' ideas.
>
> The teacher walks from group to group, quietly working to help them. He patiently works with some students who struggle more than others, pointing to words and bolded headings in the data sections they were supposed to read.
>
> Once groups have considered possible answers to the two questions, and what they find interesting in the data, the teacher has each group write their work on the whiteboard.

> One student is reluctant and asks "Do we have to answer them?" "Yes," the teacher replies. The student responds that she doesn't like math, and, "This part is too hard." The teacher tries to persuade her that "the class is trying to do something different" through examining data sets that highlight violent crimes. This is an important issue in [the neighborhood] and he wants them to consider the possible causes.

There are two instances in this sample that make it expressive. First is that students are using their journals to formulate and write down ideas, and second, when a student comments on dislike of mathematics, the teacher responds that the class is trying to mathematically understand a serious community issue.

Another mode of discourse is persuasive. Persuasive discourse often takes the form of advertisements, political propaganda, geometric proofs, trigonometric identities or relations (whether true or in-the-process of being proved), declarative axioms, lemmas or theorems, and mathematical argumentation and generalization. Ultimately, the goal is to elicit specific action such as believing something is true (or false). Previously, we have seen preliminary samples of persuasive discourse in Figures 6 and 9. Here, the idea was to mathematically convince others that the products (the paragraph on poverty; graphs of longitudinal minimum-wage data) can support the issues under consideration through arithmetic and visual representation. Consider a further sample taken from Heinz von Foester High School's students.

> The teacher reintroduces some data from a previous lesson that looks at the TB/HIV rates, and asks students to work with this data from their [mathematics journal] books. Gwen emphasizes that this will also be a tutorial using the TI-83s. A transparency with the following table (data from the WHO documenting the number of TB/HIV incidences in several regions) is placed on the OHP:

Table 7

Data from the World Health Organization on the Number of TB/HIV Incidences

Y	39	47	5404	28432	34705	90881	178,858	309,088	852,266
X	0	1	2	3	4	5	6	7	8

> The teacher demonstrates how to use the "Stat" function on the overhead TI. Some students struggle but they get help from others in their group. The objective is to use the "stat" > "calc" functions to symbolically represent the exponential equation. After several students present their work on the OHP, the teacher models the equation as $y = .24.76 * 3.26^x$. Discussion continues around interpreting what a, b mean. Students plot the correct equation and consider what might fit as an appropriate model of the data and why, and on what domain it might be best modeled. The teacher comments how the students will be the future scientists who will be creating these mathematical models. (October 7, 2010)

This sample is highly persuasive in several ways. First, there is discussion as to how to best model the data through technology and exponential functions, and second, at the end of the lesson, the teacher shows caring persuasiveness in encouraging students about their futures. This sample can also be seen as persuasive as it involves several of the students at the OHP, trying to convince their peers why their solution is the best model for the table data.

How to make something happen. Perhaps this action could be a result of action, however, I feel it fits well with the other Actions in that from the event and meaning generated in the Critical Mathematical Literacy Narrative, this particular action creates, or enables the opportunity for, awareness vis-à-vis mathematics. For the most part, this action was evident in classroom observations with Owen and Jack; however, this action only manifested itself in Gwen's work through interviews.

In Owen's class, this Action was embedded in topics such as reasons to participate in boycotts, why petitions might be created or needed, and the worker rights of electricians, welders, plumbers, and other skilled tradesmen. An interesting class discussion from Jack's class started on pollution and expanded to:

> Discussion [about what is] seen as recyclable, compost, reusable, or hazardous. The class seeks to define "recyclable" as make into new things; "compost" as make into soil or fertilizer; "reusable" as used already, and used again; and "garbage" as landfill dump. There is also discussion on what happens to recyclables/garbage, and students use clickers to decide which of these options is the best way to deal with garbage. Results from the clickers are projected on the Smart Board. The largest percentage goes to recycling. After some group discussion, group leaders approach the board and write their responses "to prove" why they believe their answer is the best.

Other proposals form Jack's class involved student-proposed solutions to gun violence, and strategic use of homework and journal entries that answer questions and summarize information that was discovered in the day's lesson, which also challenges students to take action based on what they learned about the issue and "make it better."

> Gwen's interview data presented this conversation on How to Make Something Happen.
>
> **Gwen:** You have to have knowledge of current events and know what's going on in [y]our neighborhood, city, and the kind of concentric circles going out from here. Our district, state … I'd call it community knowledge.
>
> Who are the leaders, what are the issues, and how do they differ from other neighborhoods, [for example] … with school-enrollment numbers. Having students look at the public-school enrollment numbers for the past years compared to a school over on the [other part of town] but not asking them to consider why it is happening—just what's happening.
>
> **Michael:** Hmm. It sounds like you're speaking about a kind of knowledge about forming questions or interacting with events within the greater community.
>
> **Gwen:** Making observations. Sometimes we're also starting from a statement or an idea that we want to challenge. [For example] Is it true that [local school] has the fastest declining enrollment and then should it be closed? So let's look at whether that's true. You might say it should close, but maybe that's not the reason. [Teachers] need to engage in [our] own experiences or classrooms or situations where your students have been challenged [and consider] if oppression has been the result of that.

Here, Gwen is speaking about the need for both teacher and student to be prepared mathematically for addressing relevant social issues, whether through mathematical or sociopolitical means. This is the heart of how to make something happen.

Creating "answer space." Recall that what has been discussed so far are the various kinds of actions positioned in the narrative. Answer spaces are further actions in that they are ontological in nature. Like the other parts discussed so far that are ontological in nature, it is through such spaces that teachers, students, and individuals become who we are. These spaces shape the nature and method of what is discussed and the means to participate in such a discussion. In and of themselves, various answer spaces were evident in the research data. Gwen suggested during one interview that answer spaces are "skeletons for kids to hang ideas upon," and that instead of —in the mathematics class—just making observations about data or what the numbers are doing, providing such a space for students "gives them a way of saying … I'm going to write about this; I'm going to have to think about this and talk about it with my peers" (Gwen Cooper, October 14, Follow-up Interview, 2010). That is, for the teacher it becomes a space for organizing what they are teaching and looking for opportunities to expand on that, whereas for students, an answer space can help to conceptualize mathematical abstractions and then organize knowledge into tables, equations, or words.

It is important to create the skeletons for students to hang ideas on, as without such structures our students may not engage or connect the mathematics beyond the numbers. As Owen put it, "in my opinion these connections need to flow from the students. … [Otherwise] they [simply] offer answers and the teacher simply responds." For both Owen and Gwen it was also important not to focus on a right answer but to ask, "What does that mean?" and, "What else would you want to know about this information, data, or concept?" to arrive at an answer through a collaborative student-centered process. In my observations I commented how "creating answer space begins to open space for discussion" and provides students with the opportunity to engage in thought, reflection, and "explanation and generalization of [their] mathematical arguments." Finally, this category was not coded in any of Jack's data. Perhaps Jack was creating answer space differently from Owen and Gwen. This will be addressed in the discussion section.

Focusing on difference. This last action was evident across all three participants, but in different ways. In Owen's classes, the process of focusing on difference was pragmatically used with students to examine differences between educational-income disparities, what percentage of various ethnicities do and do not graduate from high school and why, and conjecture about why nationwide incomes range vastly between men and women in the majority of occupations. In this sense, it was important for Owen's students to "get at topics and ideas more [socially] relevant than by [Owen] just making up [arbitrary] word problems and asking students to perform algorithms." Thus, by focusing or contrasting differences, students were able to make more constructive use of applying mathematics to actual ideas and opinions.

Gwen's process was to apply the action of focusing on difference in the setting of her calculus class. As we have previous seen with her precalculus class, in Figure 8 (gender across years), this idea of focusing on difference was further expanded during a lesson with her calculus students. In repositioning the idea for her calculus students, Gwen chose to have her students understand the derivative function symbolically and linguistically. To consider such differences, questions were posed such as "What do we notice about the comparative rates of change among different groups," and, "What do you observe about rates of change for different years in a single group?" Students were able to mathematically justify observations by comparing differences through demonstrating and explaining their use of the derivative function.

Jack's use of focusing on difference was also meaningful in guiding students to consider and compare differences through mathematical means. In fact, compared to Owen and Gwen, data from Jack's classroom generated six instances of this action. Here, I will briefly discuss the purpose of his use of the action. Jack was primarily motivated to get students to compare differences in situations they may already be familiar with—through siblings, family, or participation in society. For example, using Table 5 (child wages), an algebra class considered differences such as which group of people had the lowest/highest income and used mathematics and group discussion to explain some possible reasons of the difference.

Table 8

Child Wages: Average Income after Tax by Economic Family Types (2004 to 2008)

	$ Constant 2008				
	2004	2005	2006	2007	2008
Economic families, 2 people or more	68,200	69,100	70,700	73,500	74,600
Nonelderly families	71,400	72,100	74,000	76,800	78,000
2-parent families with children	79,600	77,900	79,900	83,900	84,900
Lone-parent families	36,600	41,400	42,100	42,700	43,700
Male lone-parent families	48,100	54,900	57,000	53,300	54,200
Female lone-parent families	34,000	38,400	38,700	40,400	41,300
Unattached individuals	28,300	28,800	29,800	30,500	31,000
Elderly male	28,000	28,800	29,100	31,800	32,900
Elderly female	25,400	24,700	26,700	26,300	26,800
Nonelderly male	30,700	31,900	33,200	33,500	34,400
Nonelderly female	26,500	26,800	27,200	28,400	28,300

Another such lesson was with older students and situated in the purchase of a new car and the information provided by the curriculum (see Figure 11).

Types of New Cars

Financing Terms and Rates	
36 months	4.25%
60 months	3.8%

Ford Focus		Honda Accord EX	
MSRP	$19,800	MSRP	$28,490.00
Options and Accessories		Options and Accessories	
Heated Front Seats	$0.00	Protection Package	$375.00
16" alloy wheels	$0.00	Leather Upholstery	$0.00
SYNC communication and entertainment system	$0.00	Honda Plus Warranty	$888.00
		Freight	$1,550.00
		Air Conditioner	$100.00
Freight	$1,350.00		
Air Conditioning	$100.00		
Discounts	$4,400.00		

Figure 11. Focusing on difference

Students were to decide on a purchase based on prior work, taking into account budgeting and other problems such as credit loans and purchasing power. Overall, the goals were to understand, from a daily quantitative-encounter perspective, costs and reasons why financing is more expensive than purchasing, and to think critically about all these issues.

Results of action. This last component may also be regarded as the last kind of context discussed in the above sections. The results of action position the narrative into two outcomes: inductive and proletarian. Briefly, inductive results of action are based on observations made by students, the teacher, or a collaboration of the two. This type of result describes patterns, functions, properties, or statements based on observations from prior experience. Proletarian results of action are based on something important to the individual such as race, heritage, SECS, or something of an ideological or political nature. Proletarian results are "folk" results in that the individual has created a result that has significant self-meaning.

Inductive results of action. As manifested in the data, inductive results of action are fewer in coded instances than proletarian results of action. This may be due to the nature of the study and will be further examined in the discussion section. In the meantime, I will discuss two interesting inductive results from Owen's and Gwen's instruction, although the theme was also present in Jack's instruction.

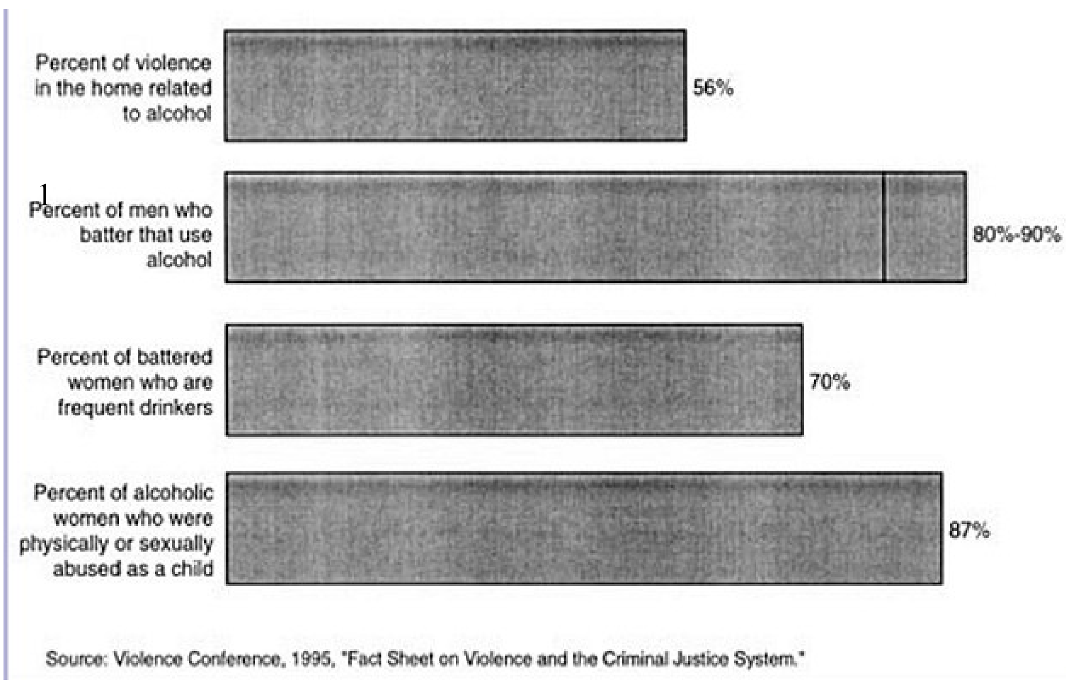


Figure 12. Relationship of alcohol to domestic violence in San Mateo County

Inductive results from Owen's instruction have briefly been highlighted in Figure 12, borrowed from a national campaign against domestic violence, and in Figure 14. These data were also part of a community presentation against drunk driving, conducted by the school.

Inductive results of action from Gwen's classes were situated in the lessons on TB that were previously discussed in the social Bardo section. The inductive results during this lesson could be seen once her students tried to find the optimal placement of sleeping mats. Recall the restrictions on determining the placement of mats: they needed to be 50 cm from a wall and another mat and were 2 m x 0.75 m (see Figure 13).

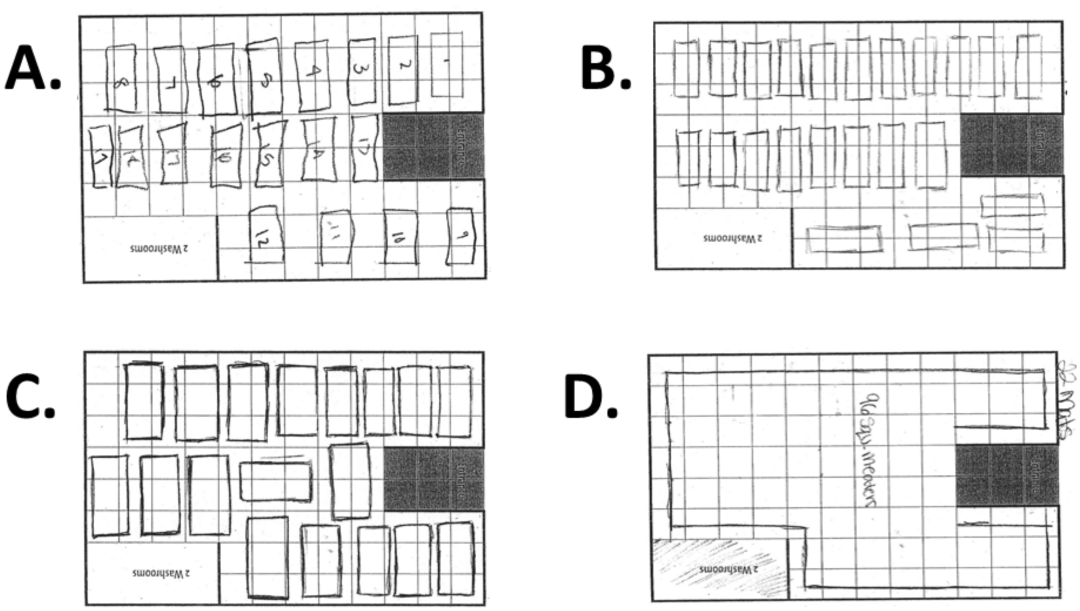


Figure 13. TB-mat placement

In working on the problem, students observed that was difficult to be 100% certain which of their peer's mat placement was optimal. We can see in Figure 13 that several of the presented solutions were similar in their understanding of the problem. However, it is clear in diagram B that the solution disregarded the space restrictions. What students were certain of was that by figuring out the area of the shelter floor and a number for the space given to each person, they could then proceed, perhaps arbitrarily, filling in the space needed for mats. It is interesting that in Diagram D, students incorporated the restrictions by creating a barrier 50 cm from the wall, which would potentially influence the mats' placement.

Proletarian results of action. As briefly mentioned, proletarian results of action are considered "folk" results in that their meaning—social, political, or otherwise—has deep personal significance to the individual. Owen and Jack had the highest number of coded incidences for this category. Gwen's use of proletarian results of action was essentially captured in three examples of protest against the local school board and organizing students in a march to create awareness. Owen's samples were also very political in nature but entirely produced by his classes or student groups, not teacher driven as with Gwen's work. Jack's samples were less political (in motivating students to resist dominant thinking and engage in transformative social action) and more about facilitating students' mathematical skills and knowledge for becoming active citizens. Examples of proletarian results of action from Owen's classes were quite political in nature. Previous examples exhibited in Figures 7 and 14. These figures illustrate the class' concern with the issue of racism that influencing a jury's guilty verdict in capital convictions.

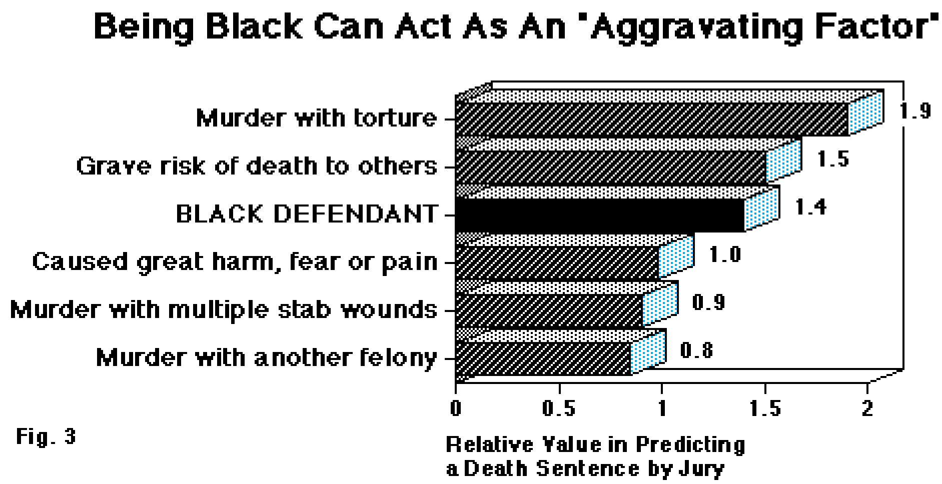


Figure 14. Ethnicity and capital conviction

Although this graphical representation highlights how "Being African-American can act as an aggravating factor" in such verdicts, it should be noted that the teacher's role was in facilitating where and how to find information, developing arithmetic competencies around proportional reasoning, and positioning the sociopolitical aspects in the Critical Mathematica-Literacy Narrative. Samples from Owen's instruction included evaluating the relationship of an officer's rank, the number of deaths during the Vietnam War, and drawing on consequences of the Afghan and Iraq Wars. Students also spent time answering and dissecting questions about government spending on these two wars and projected spending estimates on the new healthcare plan.

In contrast to Owen's students' proletarian results, which involved political considerations, Jack's efforts were guided by social and citizen values he wanted students to gain from his instruction and to both think critically and use, or expect to use, everyday mathematics tools to participate in a democratic culture and awareness in potential change. Arguments for this point are reinforced in two previously exhibited samples in Table 5 and Figure 5. Examining the nature of Jack's influence on proletarian results suggested he interprets the role differently from Owen, whose objective was more political. During the lesson in which the context involved Table 5, two particular questions posed to the class resulted in discussion: Why do companies use child labor and why do families permit their children to participate in it?

However, not all results were teacher generated. When there students were determining the amount of trash and recyclables generated by the school, the items were counted, weighed, and compared. Table 9 shows some of the results.

Students were to create physical representations for calculations they found. For example, one group calculated 35% of the total trash as food waste, whereas another group calculated 33% as compostable material. In particular, each student also brought in materials such as plastic bottles, cardboard, plastic bags, and grocery-store advertisement flyers to demonstrate how much was wasted.

Finally, I would like to briefly highlight the role of proliterean results in Gwen's instruction. Data coded from observations and interviews with her did not indicate the political or democratic-valued approaches of the other two participants. Gwen's incidences of these results were balanced between student and teacher and largely generated around contexts that were of particular interest to students. In one example, she started a lesson with the following:

> "Kids! Listen up! I just found out that, in 1998, more restaurant workers were murdered on the job than police officers. It's unbelieveable!" To this, several students cried out, "Is this TRUE?"
>
> "I like your question. We'll have to investigate how to decide if it's true." The class then discusses some reasons why the homocide rate might be higher for fast-food workers than police officers. Gwen highlights one group's proposal that perhaps there is a connection to location: "What if the location of McDonald's and similar restaurants could be in high-poverty neighborhoods? How would this shape the homicide rate?"

Table 9

Waste Audit Information

Waste categories		From garbage bags: Bag weight kgs (for 1 day)	From garbage bags: Approx. kgs/ school year	From recycling bins: Bag weight kgs (for 1 day)	From recycling bins: Approx. kgs/ school year
Recyclables	glass bottles/jars	0.35		2.4	
	metal cans (pop cans) & rigid containers	0.26		1.45	
	plastic bottles & jugs	0.55		0.8	
	plastic water bottles	0.45		3.6	
	juice boxes & milk cartons	0.4		0.4	
	styrofoam packaging	0.5		0.2	
	plastic bags	1.95		0.5	
	classroom/office paper	1.35		45.5	
	magazines/flyers/new spapers	2.6		104	
	paperbags/bristol board/cardboard	1.35		8.1	
Compostable	food scraps	24.45		0.8	
	paper towels & tissues	6.5		0.2	
Reusable	printer & toner cartridges	0		0	
	reuseable things	2		0.05	
Real garbage	real garbage	18.55		1.03	
	hazardous waste	0.1		0	

> Students brainstorm to determine if fast-food restauruants and poverty rates within local neighborhoods have any connection. Other discussions involve how fast-food workers (especially younger workers) may be completely unprepared in how to respond to a threat with a weapon and how this may contribute to the higher homicide rate. (October 29, Classroom Observation at Heinz von Foester High School, 2010)

Teacher structures. Teacher structures consist of two parts: didactical and coping. Didactical teacher structures include things such as professional obligations, available resources, peer recognition, and challenges faced by the critical mathematics educator. Coping teacher structures address how teachers make due during their working life, processes, and people who are part of the course of coping, and accompanying responses.

Didactical. Didactical teacher structures include professional obligations, available resources, peer recognition, and challenges the critical mathematics educator faces. Data suggested participants draw on various concepts when considering professional obligations. During early observations participants tended to think of professional obligations in terms of what was expected from regional mathematics educators associations. For example, Owen cited the NCTM several times, as a place he turned to for support. He specifically talked about the equity principle as influencing how he saw his obligations and his reading of *Focus in high school mathematics: Reasoning and sense making in teaching statistics and probability* (National Council of Teachers of Mathematics, 2009). For Gwen and Jack it was more important to keep active in such organizations by attending annual conferences or meetings. Because all participants were veteran teachers, it is not surprising they appreciated to such opportunities. Realizing one's professional obligations may contribute to a teacher's long-term success and pose difficulty for new teachers. Owen commented,

> New teachers have all these classes and education and degrees. [Yet] everybody says we need to support the kids and make it so they can succeed, but what about our new

> teachers? Why can we not support them to become better? (Owen Harper, June 25th, 2010)

It appears that teacher educators are not advocating for inexperienced teachers to actively engage in obligations beyond the school and to take full advantage of professional associations. However, this is even more important for a teacher engaging in CML. Participation through organizing obligations is just a starting point and is both a struggle and answer. Teachers are pushed—blamed for academically hurting students, and blamed for broad systemic failures; teacher unions create obstacles for proper reform (Greenwald, 2011)—when we need to be fluent with the tools of the trade but also flexible enough to improvise in particular moments.

It poses questions of available resources for the critical mathematics educator. Professional obligations are established at regional or national levels, whereas resources vary from teacher to teacher. Resources address our responsibility to be knowledgeable about "curriculum expectations, big ideas addressed by the expectations, lesson goals, prior student knowledge, possible student misconceptions, and what future lessons may focus on" (Jack Harkness, December 7th, 2010).

Coping. Coping teacher structures come to the forefront during, and after, the teacher's professional day. These structures include the processes and people who are part of the course of coping. Data indicate different kinds of coping methods. As we are still in the ontological spaces, these are the existing methods CML teachers use to cope and address stressors of the job. In fact, kinds of coping only appeared in Owen's school. In the classroom he had positioned a coffee station where anyone could get some of the coffee. It was clear Owen was trying his best to encourage student to be mature and make their learning environment less threatening. His class followed "the no complaining rule" and tried to find positive ways to deal with negativity. These coping methods require a listening person such as a spouse, peer, co-teacher, or friend. For a brief moment, I assumed that role during a school assembly:

> The assembly continues, administrators say grief counselors will be available to all faculty, staff, and students. This is because two nights ago, a student committed suicide. This past year there have been sixteen student suicides. Owen comments that he has stopped going to student funerals. (Owen Harper, July 11th, 2010)

This is the only instance of such dissociation among the study participants. Owen's comments capture both the limits and boundaries of caring put in place by the profession. Although it indicates a high level of objectivity, I wonder how students may begin to characterize such behavior, if faculty are not at a student's funeral. It also draws on the need to recognize and care for the mental health of student and teacher. Maybe, as educators, we can turn the situation around through reviewing why, during the past year, 16 student suicides did not meet adequate attention from administrators.

The Nature of CML Teachers' Instruction Aligned With a CML Philosophy

Findings from interviews with participants suggest means by which they align their pedagogy with a critical mathematical literacy perspective can be outlined among four kinds of value interpretations participants apply when conceptualizing their CML practices. The table below highlights the organization of this second part of the results. In terms of this second part, the first kind of value interpretation is epistemological. During interviews about teacher beliefs on the nature of their own ideology and on CML, discussion of epistemology was categorized into utilitarian, purist, and social change. A second kind of value interpretation dealt with the teacher's interpretation of the role of mathematical knowledge. There are two categories here: functional and organic. Also reflected in participant interviews is the third kind of value interpretation arguing for the participant's interpretation of the sociomathematical role of the student. This was categorized as active or passive. Finally, the teacher's value interpretation of the culture of CML is organized into four divisions: sociological, ideological, sentimental, and technological.

Table 10

Outline of the Second Part of the Results Section

What is the nature of CML high school teachers' instruction aligned with a CML philosophy?

Value interpretation	Category	Attributes
Epistemology	Utilitarian	Participant focuses on promoting procedural fluency or recognized a need to produce efficient and productive workers.
	Purist	Participant's epistemology is situated in the everyday skills and practices students need to participate in society further discussing the importance of reinventing realistic problems.
	Social change	Participant's epistemology focuses on learning and activity to empower students with knowledge to make connections between critical knowledge and possibilities for personal and social transformation.
Role of mathematical knowledge	Functional	The participant states that the role of mathematical knowledge is cumulative, transmitted based on the needs of society not the individual, useful in technical understanding.
	Organic	The participant comments that his or her beliefs on the role of mathematical knowledge is actualized by the student's culture, process-based, and should move beyond algorithmic and procedural fluency.
Social mathematical role of the student	Active	A perspective that reflects beliefs that the social mathematical role of the student is to use mathematics in ways that promote public debate, scrutiny, and critical questioning.
	Passive	A perspective that reflects beliefs that the social mathematical role of the student is passive in that the student should use mathematics in ways that support, establish, or reinforce a status quo.

Value interpretation of CML	Ideological	Beliefs that the form and content of the culture of CML teaching is ideological in nature; that is, it is useful as a means to produce students with similar values, beliefs, and philosophies.
	Sentimental	Beliefs that the form and content of the culture of CML teaching is sentimental; it is composed of feelings concerning people and behavior.
	Sociological	Beliefs that the form and content of the culture of CML teaching is sociological in nature; that is, it is composed of customs, institutions, and rules and patterns of interpersonal behavior.
	Technological	Beliefs that the form and content of the culture of CML teaching is technological in nature; that is, it is concerned with the manufacture and use of tools and implements.

Epistemology. An epistemological perspective on the nature of CML instruction emerged may be organized as utilitarian, purist, or social change. In general, a utilitarian perspective focuses on promoting procedural fluency or recognized a need to produce efficient and productive workers; a purist perspective values reinventing realistic problems; and, that of social change positions a view of knowledge as relevant in connecting critical knowledge and possibilities for social transformation.

***Utilitarian*.** Mathematics educators, most of the time, focus on teaching skills (exhibited as problems A1 and A2 in Appendix E1 and E2, from a document passed out during a professional development Jack attended on December 16, 2010), not teaching citizenship, or how to even apply mathematical literacy toward becoming a better citizen. The value position that "industry wants [mathematics educators] to create these types of workers" (Interview with Owen Harper , June 5th, 2010) captures a utilitarian epistemology that focuses on procedural fluency and places pedagogical emphasis on the teacher as a vehicle for producing efficient and productive workers. This view is teacher centered and, compared to the other two epistemological values, is positioned as the most traditional. It is perhaps the belief(s) that needs changing most, to be pragmatic as a CML teacher. This belief holds that mathematical knowledge is axiomatic in nature and, in focusing on mathematical efficiency and skills, is not overtly concerned with the different ways students are best able to practice mathematics. As Owen put it, these beliefs are best interpreted as "believing that mathematics primarily is used to set up arguments and proof—that, done correctly, math arguments follow logically" (June 5, 2010).

This viewpoint positions classroom norms and discussion from, or as centered on, persuasive discourse, similarly discussed on pp. 53–54; persuasive in the sense that instruction is for the sake of learning mathematics—not to engage in critique or analysis of sociopolitical issues. Clinchy (1996) provided additional commentary that utilitarian epistemology would lead teachers to insist students justify every statement, instruct in ways that encourage finding flaws in reasoning, and ensure what is presented meets with criteria already established by the community. As a criticism of this epistemology, from a CML perspective, the educator focuses on educating students to be efficient and productive in the ways they apply mathematical literacy. Finally, it is interesting to note that among the participants, utilitarian and passive beliefs were least valued.

Purist. Participants' comments on epistemology that is situated in the everyday skills and practices students need to participate in society and commentary on the importance of (re)inventing realistic problems was categorized as purist. This view on epistemology is less tradition and more progressive than a utilitarian perspective. It is rooted primarily in a social process of mathematical discovery (Lakatos, 1976) in which teacher and student generate and critique knowledge through a process of refining mathematical understanding and is similar in nature to problem-based (Polya, 1945) and problem-posing (Lampert, 1990). The purist focuses on dialogue as a central process to constructing mathematical knowledge. The importance of dialogue in mathematics connects with the means of social negotiation on which students build and reshape their understanding of mathematics through social interaction. In fact, Brodie (2000) suggested that teachers adhering to a purist view might represent an idealized kind of teacher often pictured in discussion of reform curricula.

Participants commented on purist views, finding mathematics to be something one can use every day; that mathematics helps students see that they can solve a problem or social issue, and that engaging in that process of mathematical activity solidifies students argumentation about what goes on in society, making sense of how somebody is arguing their point. In acknowledging that mathematical activity becomes more socially-situated, it is less dependent on the needs of corporations, and becomes prioritized to acknowledge mathematical literacy, not necessary critical, in that teachers holding purist views engage students in worthy mathematical activity such that student “empathy for and sense of affiliation with mathematics together with the desire and capacity to learn more about mathematics when the opportunity arises” (P. Cobb, 2007, p. 9). Participants also discussed a more student-centered means of instruction, in that through problems and inquiry, students will see accurately when they are going to use mathematics in real life, with classroom discourse about engaging students in real-life situations and making instruction and activity meaningful. Jack suggested that provincial ministries of education (Ontario Ministry of Education, 2004, 2010; Prince George Board of Education, 2004) are applying “philosophical discussion supporting [purist-aligned] curriculum more favorably enabling support for teachers engaging students in real life situations” (Jack Harkness, November 6, 2010) and making problem-based approaches reflect problems in real life, not merely mathematics problems to be solved. The distinction is subtle but important: in positioning problems in real lived experience instead of solely a problem to be solved gradually moves toward an egalitarian model of mathematics pedagogy in which students progressively hold more input in instruction and deepened responsibility in learning. Such instruction skillfully supports dialogue and constructions of shared meaning and knowledge, at the juncture where personal and social dimensions cross (Ross, 2004), that emerges from experience.

Social change. Social change epistemology refers to teacher comments that emphasize learning and activity to empower students with knowledge to make connections between critical knowledge and possibilities for personal and social transformation. Social-change beliefs enable the teacher to include socially relevant topics by shifting their instruction to using the curriculum in ways that get students to question what they are learning. That is, guiding students down a path that they can will recognize and be positive agents of change, but also to help them understand there is a deeper meaning to issues than people normally think.

This expanding of the curriculum to enabling students to see the deeper meaning through engagement, it is also about mathematical critical thinking in seeing socially just issues as connecting to that deeper meaning. In seeking to connect with that meaning we ask "What is others' experience leading to such mathematical understanding?" Social justice is an avenue to open up students' minds to think critically while engagement is a short-term benefit: "embracing new ideas, looking for what is `right' even in positions that seem initially wrong" (Clinchy, 1996, p. 207) gets students to be agents of change by thinking critically about issues.

The social-change perspective suggests mathematics educators could use mathematics to change society, to teach in ways to encourage children in science, mathematics, history, or

English by helping them find that passion. The following dialogue highlights this part of the social-change epistemology.

> **Michael:** So, [CML teachers] should be thinking about helping kids become more than good citizens?
>
> **Owen:** Well, not just—good citizens, but more …
>
> **Michael:** Active?
>
> **Owen:** More … *involved*. We really should be thinking about getting kids to be involved members of society. That they're in there and you're not involved in something unless you're passionate about it, right? But we're also not helping them find passion when all we do is make them do worksheets out of the book every night.

This touches a challenge that mathematics education has neither properly solved or fully dealt with: career switchers and professionals, especially in the sciences, who have been recruited to teach in the secondary school. Often these new teachers *do* believe that teaching mathematics is practicing algorithms and making kids to do worksheets to reinforce class material. Perhaps, with regard to epistemological views, new practitioners from the business and professional world are not adequately prepared to teach mathematics with social-justice themes. When beliefs among career switchers are examined, early results from Lee (2011) initially suggested individuals are more motivated by current career idleness ("dissatisfaction with the previous career") or feelings of professional uselessness (individuals "had chosen the wrong career path") and not encouraging in terms of a social-change epistemology.

Teachers' interpretation of the role of mathematical knowledge

Functional. Participants' interpretation of the role of mathematical knowledge as *functional* sees the role as based on the cumulative transmission of mathematical knowledge, from teacher to students, and primarily based on the needs of corporations, industry, or society; not on the needs of the individual. A functional view also positions the teacher as the basis of the starting point of students' knowledge, and one which regards the role of mathematics as that of proof. Those who view mathematics in this way consider how to formulate an idea, thinking how to get from where they are to where they want to be. In this line of thinking, participants often spoke of how society expects mathematics education to be valid explicitly for technical understanding. Consider how a primary goal of the American Competitiveness Act (2006) was to get more engineers, scientists, and biologists into the field of mathematics teaching; a means of getting more "highly qualified" individuals into the classroom. However, there were several issues, raised by Owen, once such individuals arrived in the classroom. In citing his work with two such individuals, Owen commented "when they came to our school, to the classroom, they questioned `what do you mean my kids can't do fractions?` or `what do you mean 12th graders don't understand the meaning of a derivative?`"

The primary concern was that perhaps such uninformed pedagogical questions may eventually contribute to mathematics anxiety. Functional views on mathematical knowledge, in relation to secondary mathematics, teaches students that procedural ways of learning "skill after skill can be harmful. … [For example] students know how to calculate a percentage but they don't know when to do it" (Jack Harkness, November 6, 2010). The way this may be harmful is, for instance, if you ask students to calculate the tax on an item, they can find the price but do not understand when to apply the skills learned. This is because the learning out of context. So, functional views of mathematical knowledge, as understood in relation to secondary mathematics, may best be understood if a teacher were to use a text, sticking solely to what's outlined in the table of contents. From the CML perspective, by going straight to the mathematics, this teacher will miss an opportunity to open students' eyes to issues of injustice and social concern, and to think critically about such issues. If too many teachers do this then nothing will ever change.

Organic. The teacher's interpretation of the role of mathematical knowledge as organic reflects teacher comments that the role of mathematical knowledge is actualized by the student's culture, is process-based, and should move beyond algorithmic and procedural fluency. This view contrasts with a functional view, previously discussed, which was more teacher centered and based on the needs of society rather than those of the individual. Consider how mathematics has been culturally actualized by individuals for several millennia. Egyptians used mathematics to guide astronomical and construction issues; the hexadecimal number system of the Babylonians is seen today in our used of 360 degrees in a circle and division of an hour in 60 minutes; the idea of zero escaped mathematics until introduced into modern day Arabic numbers during the first millennia (Fenn, 2007). More recently the cultural use of mathematics has been further examined by mathematics educators (Bishop, 1988; Gutiérrez, 2002) focusing on the practices and process of mathematics.

People have always tried to understand mathematics. By being culturally actualized by individuals, I am referring to the ways mathematical understanding helps people in banking, retail, manufacturing, and industry. As an example, in one of Owen's classes, students were using AutoCAD to design a home to be presented to local community members. In discussion of building the model, conversation considered what it took to get a scale model. The class worked to find all the measurements and began to question why the fractions were so small—they were factoring models too small. One student said, "I can't do this because it won't be feasible. It would be too small to handle with my fingers. I would need tweezers." The points is that students can gain the content knowledge needed out of personal experience. They can bring it to their level and relate it to their lives. This relationship makes it more meaningful to them when they are doing mathematics because the students who are working on problems they can relate to understand its relevance, see it going on, and then attach the mathematics to it. Thus, it has more meaning and engagement because it's actually about something relevant.

Finally, in moving beyond procedural fluency, Gwen commented that "It feels pretty hollow to just have a bunch of procedures that you're teaching and then having people do them." It is important to reconcile our knowledge that there are procedural things that are useful in mathematics that need to be taught, but it seems a lot more engaging for students to practice mathematics in the context of problems, especially when those problems can be about something to which they can relate. Organically, pedagogy is shifting linearly following a curriculum, chapter after chapter, from teacher-centered instruction to one more student focused, in which students can see themselves and their relationship to the problem. The goal is that students will see that in mathematics there is something to relate to that they need to know how to do, some perspective the student might be able to identify either culturally or socially in some way. In letting the student draw out the mathematics, teachers may not recognize the benefits right away, perhaps not even in the time they are their student. Jack commented that "it's something that really does take years and years of consistent effort" from teachers to teach in less teacher-centric ways, working to get students to support their own thinking through mathematics.

Teachers' interpretation of the social mathematical role of the student

Active. Participants who valued the social-mathematical role of the student as active frequently described their CML instruction as a means of promoting public debate, scrutiny, and critical questioning. Examples of participants' valuation of the role of the student were indicated in data from Owen and Gwen. Jack's interpretation of the social role of the student is more fluid and will be discussed in the following section (undoing the damage). For Owen, "active" could be described as "tearing apart an issue" (Owen Harper, June 6, 2010) and really using mathematics to find alternatives or solutions to bolster students' conceptions of how they can support their point of view on an issue. Ideally this involves problem-based instruction, open-ended dialogue with students questioning what it means, asking what can we do about it, and where can we go? In particular, one lesson involved examining data describing reasons individuals were pulled over by police. These data were examined thinking about what mathematics is most useful and also looking at what is fair or unfair in possible results. The important ideas should be student driven. Such ideas addressed drug use on the school campus, border-crossing and immigration issues, and picketing KFC with signs showing results of data analysis.

Gwen's interpretation of the social role of the student was privileged by the structure of her school. She was able to work with the same students over a 4-year period. This positioned her to help guide students to further develop their skills in public debate and critical questioning. For example with her algebra students, in finding and interpreting the slope of a graph, and with her calculus students in working with the area under a curve, she questioned "why practice this mathematics in an abstract or symbolic way when you could take a problem with real data and come up with a solution that had meaning in that context?" (October 14, 2010).

Overall, interpreting the social role of the student as active is not to reduce mathematical concepts to which students are exposed, but rather have them question in ways that might be more memorable. So, think of the social role of students as helping children get more enjoyment and engagement, especially at more advanced levels, and be comfortable with mathematics that enables them to have rich conversations, evolving to say "can we do this more?"

Passive. The passive view on the social-mathematical role of the student has the same number of counts as utilitarian as highlighted on Table 9. Perhaps it is that the two depend on each other directly. A passive view on the social-mathematical role of the student leads to a utilitarian view of epistemology, or vice versa. I have referenced this through teacher comments that the social-mathematical role of the student should use mathematics in ways that support, establish, or reinforce the status quo. Having said that, it is not a surprise that participants did not value this kind of social mathematical role for students (as in Jack's case) or were not the focus of such value interpretation; that is when considering the combination with action and in relation to the other kinds of value interpretations. However, although participants could still acknowledge its value, having such low counts did not entirely allow it to be ta large priority in their overall beliefs and values.

The connection to utilitarian (see Table 9) suggests passive values in the context of the classroom may be the beginning stages of development of a teacher's value interpretation of the social-mathematical role of the student, beginning with passive/utilitarian and likely ending at active/social change. I conjecture that teachers, who are at a passive stage indicating such preferences to related beliefs and values, are indeed at this beginning stage of the development and understanding of the social-mathematical role of students.

Teachers' value interpretation of critical mathematical literacy culture

Ideological. The teacher's value interpretation of CML culture as ideological highlights participant comments and thoughts that the form and content of the culture of CML teaching is ideological in nature; that is, it is useful as a means to produce students with similar values, beliefs, and philosophies. Among the seven kinds of value interpretations, ideological was the densest. This is represented Table 11, which compares category count of the seven kinds of value interpretations across the three participants.

Table 11

Comparison of Category Count of Seven Kinds of Value Interpretations

		Jack	Gwen	Owen	Grounded/ totals
Epistemology	Utilitarian	1	0	6	7
	Purist	3	2	11	16
	Social Change	6	1	19	26
Teachers' interpretations of the role of mathematical knowledge	Functional	4	2	4	10
	Organic	6	3	17	26
Social mathematical role of the student	Active	0	2	17	19
	Passive	0	2	5	7
Value interpretation of culture of CML	Ideological	3	4	26	33
	Sentimental	2	17	13	32
	Sociological	3	1	15	19
	Technological	2	2	8	12
	TOTALS:	30	36	141	207

Owen had the largest number of instances of the ideological interpretation, in comparison to the other two participants. As an ideological CML teacher, participants interpreted the overall culture as encouraging students to develop their own independent views and beliefs.

Teachers' interpretation of the CML culture is an ongoing process. It is not much about handing a perspective down to students; rather it is being informed as a teacher to have enough knowledge and information about both mathematics and current affairs to be able to engage students in both forms of dialogue. Thus, students may see how and in what ways mathematics becomes more and more important to everyday life. Owen, the most ideological of the teachers, explained, “I get a lot [of ideas] from newspaper, texts, Internet … everything I come across I think, `how can I take that to my classroom? Can I make it interesting to kids?`” During one interview we discussed a news report about mineral reserves recently found in Afghanistan. What would that have to do with children?

We planned a lesson to start going through it. They’ve found mineral reserves; how much? What’s it going to take to get those reserves out of there? Who will benefit? Who in the world makes the mining equipment to do that? What are environmental and political positive and negative aspects? What is the best way to use mathematics to support one’s arguments? It is not about a teacher’s individual persuasions but about helping students to find their own. One has to be aware that they’re not reproducing their own viewpoint on policy or political ideas or religion. It’s about the students parsing the issues. Once the students comprehend how to use mathematics to address one issue, that issue can be used as a springboard to address other issues.

This view addresses mathematics, current affairs, and also the history of mathematics. That historical understanding can give teachers a better idea of how mathematics interacts with the world, looking at how mathematics was done in certain time periods: When were these concepts? Who proved them and why? Also looking back gives a broader perspective in conjunction with world events and history. Looking back, teachers can consider the mathematics done today and how it is applicable to current events throughout the world. Teachers also have a duty, in moving toward more student-generated social participation, for example when presenting something controversial, to present all sides of the issues, enabling students to justify for themselves what it is they think, feel, and want to do.

Sociological. The teachers' value interpretation of the culture of CML as sociological refers to how participants' data exhibits patterns in the form and content of the culture of CML as sociological in nature; that is, it is composed of customs, institutions, and rules and patterns of interpersonal behavior. Mathematics classroom customs and practices are rooted in a practice of proving something valid to teachers, students, or others. Teachers' professional standards suggest classroom customs should focus on perseverance, verbal cues, and recognizing and correcting mistakes (National Council of Teachers of Mathematics, 1991). Although such customs seem to be valid practices, they do not engage students to fully participate in understanding social challenges and possible solutions. Teachers must learn to practice patience, practice not responding immediately to the first answer, and admit that the teacher is not always the center of learning.

Engaging in classroom customs lets teachers consider what else is valid from a sociological perspective in the culture of CML. Teachers perform as a guide, helping students to work through the mathematics and see that it can be a tool for citizenship, for an engineer, and for changing society. Teachers need to allow student culture to become a part of the classroom. The topics in a CML classroom can be controversial, but teachers should be working to allow students to speak to each other, not trying to speak over each other. These rules and patterns for classroom discourse should allow all students to see education as an introduction to the complexities of life, the world, what we know, and where we fit over time. Mathematics may be a quest to solve problems, supporting the individual needs of students in the context of the problem, and working to understand the rules and patterns that apply to everyone. The study of mathematics should prepare students to think about their choices, working to change things for the better, and can add to the social skills students need to interact effectively with other members of society.

Sentimental. Teachers' value interpretation of CML culture as sentimental refers to comments that the form and content of the culture of CML teaching is sentimental; that is, it is composed of feelings concerning people and behavior. Referring back to Table 9, the sentimental valuation of CML was the second most grounded category across all three participants. Values as sentimental do not mean having emotional or nostalgic beliefs, but instead address a deep awareness and caring for one's students. Consider Jack's comments (December 2, 2010):

> Last year I have a student who grew up in poverty in Ecuador and she hated math and her mom hated math and transferred that right to her. When she came to my classroom, at the end of the school year, she did a presentation on poverty, had several graphs and statistics [about world poverty], understood what they meant and shared that with the class in a very heartfelt way and felt good about math perhaps for the first time in her life. I can't tell you how good that made me feel.

Here Jack commented on the personal value the student and the teacher gained by the student's presentation on poverty, a topic the student had experienced. The teacher provided the opportunity for the student to explore the topic, providing a change in the mindset of the student. A teacher who values CML culture as sentimental, the assessment of a presentation cognitively shifts students' perspective to becoming familiar with *new* forms of interpersonal behavior and learning mathematics by focusing project/problem-based learning rather than focusing on examinations.

This cognitive shift has been described by Lesh and Doerr (2003) as moving from "applied problem solving treated as a special case of traditional problem solving" to a more modern understanding in which "problem solving is treated as a special case of model-eliciting activities" (p. 4). This subtle distinction was made in the section discussing purist views on epistemology about making problems reflect realistic, everyday challenges and not simply mathematical problems to be solved for the sake of completion. This cognitive shift can be extended to introduce students to conceptual tools that include convincing ideas that reveal important characteristics about how they are understanding the problem-solving environments.

This process of mathematical self-convincing generated to better understand a problem-solving environment frames the learner as the subject ("I am working with tools, ideas, and characteristics to relate to the problem") and positions tools, ideas, and characteristics as the object for the student's knowing. As Gwen put it,

> it takes very deep dedication and professionalism. I'm hired to teach mathematics but all my kids aren't going to come to me looking, sounding the same. I've got to figure out how to teach best with each one of them.

This interpersonal professional dedication is also an attribute that sets a sentimental valuation apart from the other three. We want our students to have knowledge of mathematics and good feelings about it as well. Continuing in her dedication to her students, Gwen commented "I want them to feel like it's not awful; it's not scary; it's not abstract. It's something that they can do, it can be learned with a caring teacher and it's useful." (October 14, 2010). Along with dedication there is also enthusiasm—for the mathematics and for the students. If relationships with students are good, then students are going to learn more easily and the teacher will feel good about maintaining positive relationships and a positive learning environment. Beliefs as sentimental is about wanting students to feel successful; that they can continue in their mathematics learning at a college level, doing abstract work. Part of students being successful is that they can picture something that is not going to be purely abstract for them. Finally, sentimental beliefs hold that "the teacher is just trying to make it so that it's a little easier for students to understand as we go through problems" (Owen Harper, Follow-up interview, August 9, 2010).

Technological. Technological values refer to participant comments that the form and content of the culture of CML teaching is technological in nature; that is, it is concerned with the manufacture of tools and implements. Owen commented that his technological valuation started with the idea of what was being taught to students. "Does anybody realize how much mathematics we really need just for even a job as mechanic? Usually we think mechanic—turn the wrenches and the job's done. This is not so true any more" (June 8, 2010). As we position this comment in modern society and think about the equipment and interactions that mechanics have with machinery and tools, it has become incredibly advanced; moving into the vocation fields often involves mathematical literacy in a received form of knowledge.

Briefly this demonstrates how the technological valuation of CML culture is concerned with the use of tools and implements. Classroom instruction considers tools that are important to organizing and structuring mathematical learning. For instance, a commonality with Owen and Gwen was the organization and maintenance of student mathematics notebooks for Owen, and mathematical composition books for Gwen, as a means for students to record and arrange class notes, homework, and related materials. This use of notebooks or composition books can establish uniformity in the pedagogy. This creates the expectation that the student will always be in possession of their work; students do not need to ask for or look for, paper. In Gwen's case, the entire school had adopted the idea so a teacher may ask where the student's composition book is; as if they were to bring a textbook to class. Finally, students studying science and collecting data, need some mathematical tools or mathematical knowledge to analyze that data, to ensure the experiment will make sense. If they don't know what to do with any of the numbers they collect, then their knowledge is insufficient to analyze the data.

On the foundations of a mathematics teacher's technological beliefs, Williams (2001) suggested the foundation is rooted in capitalist values in which business and corporate interests are extended through social influence, particularly on teachers, to prepare children for professional participation in industrial society. Although it is important for teachers to prepare students for vocational positions, if that is the sole focus it narrows the mathematics curriculum through valuing mathematics as an unquestioning body of knowledge, lacking critical emphasis and thinking, and positions students' mathematical learning outcomes to be based on social training in obedience and seeing mathematics as realistic only for industry-centered knowledge appropriate for vocational certification.

Discourse and practice in the CML classroom

So far results have illustrated the means through which students are challenged by existing modes of maintaining culture and the ways self-image is revised. This is the conventionalization stage of the model of the construction of mathematical knowledge—as highlighted in Figure 1. Through the ways in which culture is transmitted in the CML classroom, the students' personal meaning—personal understanding of mathematical knowledge—is mediated and revised to reflect a self-image built through image-building ethical pedagogical tools. This self-image is garnered through mathematical practice and discourse generated through agreement with and meaning making with peers.

Reference to the Bardo state relates to the appropriation stage of the model of the construction of mathematical knowledge. At this stage, the ownership of the knowledge transitions from public to private still within the social realm. As the Bardo is a space individuals pass through, shaped by culture (cultural Bardo) and social (social Bardo) factors. The Cultural Bardo affects appropriation by influencing the student's beliefs, fears, and emotions about the ways in which new mathematical knowledge is internalized. For example, if the student has previously learned from their parents that mathematics has no value then the student will be less interested. The role of the Social Bardo is to contribute to the individual's personal understanding of mathematical knowledge as the social location changes from the social (the meaning developed in conjunction with peers) to private (how meaning personally makes sense to the student). That is to say while the Cultural Bardo is where individuals internalize meaning, the Social Bardo is how personal meaning is brought about by what occurs physically.

The ontological space of the Cultural and Social Bardos represents points at which the student solidifies a perspective on mathematics. The Cultural Bardo represents a student's movement within various classes of cultural censure of mathematics. Participants spoke of seeing their students (self-)censure depend largely on parents—as with Owen's comments—and also as a place to encourage and counter such censure through helping students re-connect with mathematics in ways that help them see mathematics in everyday realistic terms. Gwen spoke of the Cultural Bardo as a belief in one's self that when society acknowledges and encourages disapproval of mathematical competence, the teacher must step in to show students "under the surface anybody who can't do math actually isn't totally self-assured that they're smart" (Gwen Cooper, October, 14, 2010).

The social Bardo occupies the ways teachers and students comprehend resistance to society through either resistance-by-choice or resistance-not-by-choice. Resistance-by-choice was viewed as becoming the first person in a family to graduate from high school. Resistance-not-by-choice was addressed in several data points in the data as poverty, displaced institutions (e.g., prison, "The Rez"), and homelessness. It is a function of the Bardo to acknowledge that which is often unspoken, especially in the context of the secondary mathematics classroom. Through understanding the cultural and social Bardo, educators may begin showing students that equanimity is possible; that in being the first generation to graduate, that valuing mathematics for its role in helping to analyze, understand, and offer solutions to challenges facing society, then the Bardo does become an intermediate state whose outcome of liberation, of transformation, arises by passing through this immediate state, turning its negativity to virtue.

Learned negativity. Participants suggested their students' views of mathematics as a subject developed chronologically, with mathematical anxiety, phobia, and fear perceived by students at early ages. On the subject of mathematical anxiety, Ramirez and Dockweiler (1987) found it has a negative effect on mathematical performance (e.g., under high-pressure assessment) and on future decisions, such as dismissing mathematical vocations. Further discussion with participants drew on the relationship between the student and their parents. That is, if parents were not positively encouraging their child's learning of mathematics, the child would learn it is acceptable not to be mathematically literate.

It is difficult for students with mathematical anxiety to help students experience and understand their relationship to that anxiety. For the CML teacher, however, it should become an opportunity to help students transform; to recognize their own negative beliefs and reposition them in a safe learning environment. All participants spoke of their own caring for students and instruction as attempts at "undoing the damage" of previous inept mathematics teachers—those with whom students learned to associate negative beliefs with mathematics. The consequences are clear if educators do not address our students' mathematical anxiety. As these students mature and enter society, they often feel ashamed because of prior negative experiences in mathematics and exhibit self-censorship, self-doubt, and vulnerability in doing mathematics (Bibby, 2002) along with self-distrust in their own mathematical knowledge (Coben & Thumpston, 1995).

Addressing the discourse of event. The discourse of event organizes classroom discourse in the Critical Mathematical Literacy Narrative as expressive and persuasive discourse. Consider the sample of expressive discourse, where the group members' act of struggling to mathematically summarize their data in their binders and the student comments that this part is too difficult. Expressive discourse helps solidify ideas through writing and seeing the connection between violent local neighborhood-crime data as a realistic setting for mathematics. With persuasive discourse, understanding arose from students trying to persuade the class that their model equation was the best representation. Although there was some discussion among students, the teacher had the final say both in which was the best model equation and in persuading students that this kind of mathematical understanding would enable them to become the next generation of scientists.

Discourse of event. Discourse of event provides the space to develop CML in two ways: expressive and persuasive. Opportunities to provide expressive discourse, as outlined in the results section, ideally helps students make personal sense—engage in a personal critique—whether in a mathematics/activity journal or in interpersonal communication with peers and group members. This draws on the teacher–student boundary and yields more space to the student. Their expressions in vocal or written statements personalize and further mathematical understanding from seeing and interpreting a context in ways with which they are familiar; that is, through sketching diagraphs or graphs to solve problems, as seen in Figure 13. Sowa (2000) suggested that through expressive discourse ontological relations among mathematical concepts the students is working to understand become further solidified through the statements (personal or interpersonal) students make. In terms of what this means for the CML teacher, it suggests instruction should prepare and yield opportunities for students to be comfortable with expressing, in their own language, what the language of mathematics forms through symbols and terms, vocabulary, and variables used.

Persuasive discourse contributes to the discourse of event as opportunities to engage in processes of argumentation and critical mathematical thinking. When situated in the Critical Mathematical Literacy Narrative, persuasive discourse can position students to evaluate a problem through ontological means in determining a solution or even whether a solution exists. Persuasive discourse is a moment in instruction when a distinction is called upon between procedural and conceptual knowledge. Examples from data highlighted in Gwen's class focused on trying to convince oneself, via modeling data on the TI calculator, of which exponential function best represented the given data.

Across all data there were 26 instances of persuasive discourse where such discourse was used to convince someone of a solution to a problem. Critical mathematical thinking occurs when acknowledging that using both procedural and conceptual mathematical knowledge contribute to a problem's solution. One can see, for example, in modeling an equation, that its graphical representation does approximately fit the given data; however, students need to look back on prior mathematical experience to be able to piece together their own questions and see what is missing; that is, to look for deeper meaning themselves rather than depending on the teacher for guided meaning. For CML teachers, this means being aware that lessons can be heavily contextualized in a specific problem—valuing proficiency and not permitting students to take their understanding to a more abstract, concept-based level and translate contextualized understanding to a more general, broader case. So, for example, in a lesson on measurement, students may be able to measure various walls and plot that data but may have difficulty applying that to measuring something very large, like a skyscraper, or very small, like a human cell. We need to take that understanding they have of mathematics in a specific problem and be able to generalize it or resolve a possible solution in a second, different case, while also being able to compare what, why, and how understanding is different by creating the concepts we are seeking.

The Teacher

Attributes of "the teacher." There were several attributes encountered in the data of a CML teacher. The first attribute of an effective teacher is one who provides several opportunities for contemplative self-analysis both teacher created and student–teacher created. One way in which participants promoted contemplative self-analysis in a teacher-created means was by critical review of a student's performance on an assessment. For example, in reviewing an assessment with her students, Gwen had students question their mistakes with considerations such as "Was this a small mistake?" To correct that misunderstanding the student could further detail "I know it well but goofed," or "I need to work on this and will seek tutoring." Student–teacher-created contemplative self-analysis largely took the shape of discussion around creating rubrics and understanding what expectations were, mathematical and otherwise, for a project. Owen discussed one such rubric where "written on the whiteboard were two columns. One was showing the mathematics and the other was writing out the steps they went through. … [These] were done by us here in class" (June 25, 2010). Jack also used similar student–teacher generated rubrics.

Another attribute encountered in the data concerns the idea of discouraging rapid reward. By this, I am referring to promoting student thinking that goes beyond the right answer. Perhaps to some degree this requires a shift in our students' cognitive perspective. Consider what is lost when the mathematics teacher positively responds to that student who answers our questions first. The need for additional understanding, for more thinking stops there. Our options are to restate question–answer and perhaps offer another effortless question and continue the lesson. Across several instances, participants discouraged this "right answer" kind of thinking. Primarily this was accomplished by the participant when students sought approval on a ventured answer. As a means of instruction, this is a very student-focused way to teach such that teachers no longer promote fill-in-the-blank responses to a text but instead say to students "OK. You've come up with a solution. What are some of implications? How can you show me it makes sense mathematically?" Thus, CML teachers need to learn not to immediately respond to the student closest to us (should we be at the front of the room); that is not to say we cannot reward the initial response, but we need to encourage peer dialogue, debate, and critique. We need to focus on these students who "look like they may answer, and often do, any question; that look like they are thinking" (Jack Harkness, November 17, 2010) but are really just waiting for the right answer and want to move on.

Teacher voice refers to an attribute of the teacher that moderates a teacher's internal voice. For example, Gwen referred to "constantly being aware of the fear [of mathematics] level" (Gwen Cooper, October 7, 2010). Consider as this fear rises, learning goes down. In the most advanced class of each participant, field observations highlighted how participants wanted to instill in their students the ideas that they, even the most fearful student, were good at mathematical pursuits; that they can trust the teacher about the simplicity of it; and that when they are working with mathematics, now in the class, later with another teacher or parent, or in the future perhaps at work or in college, they can calm themselves by remembering the big ideas. Teacher voice is also this calmness, soothing students; placed there by the teacher by developing personal relationships and trust with students so they may keep learning. When schoolwork gets difficult and they become scared or disoriented or disbelieving, previous experience with a caring mathematics teacher will reassure them that they can reach achieve understanding. In interviews about the teacher voice, Jack referred to it as the internal coach asking "What went well? What changes can be made? What are my next steps?" In this sense, teacher voice is an internal coach teachers pass on to students and equally mature teachers' own sense of what it is like to develop that teacher's voice.

Pragmatic educator. The image of the CML teacher as educator is the last attribute of the teacher I will discuss. A pragmatic view will see the importance and possibility for social change, that students' everyday lived experiences have meaning and can provide meaning-making contexts. The pragmatic mathematics educator understands that applied and realistic problems stem from these everyday life and work environments. By acknowledging the contemporary, we are also acknowledging the historical, activism and counterculture, factual, and popular culture. These environments were explored in the discussion of a Critical Mathematical Literacy Narrative's Context. This pragmatism has several roots across curricular theory and more specifically in the view of the practice of mathematics.

The pragmatic educator has ideological roots in the philosophy of science education (Popper, 1959), which suggests the scientific method is an invaluable tool in the creation of new content knowledge. In the arena of mathematics education, such empiricist views on new content knowledge deal with recognizing that whenever we claim to mathematically know something it becomes revisable, once we learn a specialized way to represent that knowledge. For example, consider the beginning algebra student who has been told by teachers that all linear equations can only be written as $y = mx + b$. When one day our beginning algebra student learns from a peer that linear equations can also be written as $ax + by = c$, the beginning algebra student is still justified in the original belief and now has support for a new representation of that knowledge. As we get into more advanced mathematical knowledge the pragmatic teacher works with students to revise their knowledge and consider how the truthfulness of something is established by convincing others that a concept is mathematically valid. Participants describe their adherence to this attribute of the teacher as opening up student learning to the value of discussion and peer critique by accepting that students and teachers are sources of knowledge and that doing mathematics becomes more than filling-in-the-blanks and incorporates expressive and persuasive discourse to convince self and other about mathematical truthfulness.

Discussion

This section will address means by which the research presented in this study is relevant to both ontological and ideological perspectives on the practice of teaching CML (ontological) and toward its culture. Recall, as discussed in the first section of the Results, that characteristic spaces emerged in which teachers and students developed relations, engaged in sociopolitical and cultural inquiry, and placed meaning of being in a cycle of analysis and critique arrived at through mathematical literacy. The second part of the discussion will consider how results presented as an ideology for mathematics education can further shape secondary mathematics instruction compatible with teacher values and beliefs conceived as foundational to critical mathematical literacy. Finally, the discussion will close with directions for continued inquiry and practice.

Understanding CML Teaching in Ontological Spaces

The results presented in the first ontological space, ways to transmit culture, highlight classroom and school practices in which students begin to believe and participate in; beliefs and participation further rewarded and recognized by administration and teachers. Continuing this tradition of maintaining and valuing dominant culture, over one in which plurality and promotion of individual and social respect is encouraged, needs to be reduced and reconsidered. The mathematics classroom can begin to chip away at the maintenance and transmission of dominant culture; yet, for this to happen takes a very dedicated and compassionate educator. It takes continued development of the teacher's practice to, as Barta and Brenner (2009) point out, highlight, contrast, and purposefully expand our limited and biased views of didactics of mathematics education to one in which it is both a tool for inquiry and a model for engaged democratic and political dialogue.

In the classroom we need to move away from irrelevant food math to a discipline that encourages broad questioning of how, what, and why society and mathematics connect. The six domains of critical mathematical literacy are but one means to effect social change with students and classroom. Incorporating mathematics problems (for example, as exhibited in Figures 7 – 11 and in Table 6 and 7) change what it means to *use* mathematics and position its knowledge as something more than what students learn in school or passively reposition from classroom setting to occupational setting.

Transitioning from illiteracy to literacy through the Bardo. In further considering the ontology of spaces of existence, the results considered the social and cultural space specific to progression from CML illiteracy to a developed literacy. That is, the space of the Bardo encases an individual's journey. From the ontological sense, the individual's being or notion of what it means to be literate from a CML perspective transitions and evolves as the mathematics teacher orchestrates the students' travel from (mathematical) death to (mathematical) rebirth. The moment of (mathematical) "death" of the student is the beginning of the Bardo journey.

This application to the classroom highlights the Cultural and Social Bardo realms. The Cultural Bardo was exhibited through participants helping students to reconnect with mathematics through positive and personal means. Within our classrooms this means educators have influence, voice, responsibilities toward socializing students to solve problems and work collaboratively. Consider this as a way to guide your classroom discussion toward having students develop *confident answers*. That is, not one of right or wrong but that focuses on the process of arriving at a *confident answer*; a solution that could be right or wrong but intrinsically grounded in he students own conceptual/contextual understanding.

The transition through the Bardo toward a more critically mathematical literate individual, in the Cultural Bardo realm, exhibits reconnecting with mathematics from more direct and personal means. In the mathematics classroom, the Social Bardo layers a political, and sometimes controversial, influence of the mathematics classroom as structuring, or creating awareness among classrooms and society, in engaging about what is means to encompass a socially just problem through math understanding. This relationship exhibited in mathematics instruction is an entry point for crossing borders between generating mathematical understanding and positive social action that may be created or acknowledged.

Strategies for addressing this varied among participants. Owen included themes such as poverty (on local Native American Reservations and in developing nations) and discussion of "what it means to be the first generation in your family to make it past 8th grade" as entry points for localizing data, creating opportunities for his students to complete writing prompts and design graphs and posters about poverty, and for overcoming socioeconomic and cultural barriers in moving out of poverty, homelessness, a Reservation, prison, and turning negative social norms surrounding these into positive means for understanding society's reactions and, more appropriately, his own students' heritage and indigenous traditions. Gwen used topics such as homelessness and homeless shelters, refugee camps, and consequences of inadequate health care as means for developing her students' positive engagement with math and relevant political events. Jack promoted the general literacy development of all of his department teachers by highlighting, during a professional development session, how varied sociopolitical topics are supported as means for mathematical literacy by standards of practice used by his Ministry of Education and that by math teachers not implementing such topics as entry points into further developing students mathematical literacy, teachers were effectively not meeting the needs of all their students.

CML Narratives. As an ontological space *Narratives* can be discussed as a more direct experience between teacher-student and what is generally regarded as a secondary mathematics curriculum. One of the challenges of teaching for critically literate mathematics understanding is that we are not always supported in our endeavors by curricula. In outlining such concerns, *Narratives*, develop the sociopolitical through Contexts, Actions, and Results of Actions. With regard to *Narrative contexts*, examples collected from participants tended to be situated in data that supports visual (as in Figures 6, 7, 10, and 12) or numeric (as in figure 11 and Tables 3-5) comparison. While there were also other contexts highlighted, this suggests participants found examples from their existing curricula lacking n that participants needed to develop their own meaningful contexts. I would suggest that future curricula should be expanded to build on these but should also consider locating meaning in additional areas not solely in that of a data set.

The participants use of *Narrative* actions formed much of students understanding as not simply right or wrong but as involving possible solutions negotiated among the class and between students. Recently in the *Occupy Wall Street* (Occupy Wall Street, 2011) discussion, citizens brought to the forefront of national discourse issues of income disparity; similarly, Gwen examined, more specifically and from a mathematical nature, how a corporation's CEO pay compared to the rest of the company's employees. Strategies that Owen used included familiarizing students with locally available crime data and probing results in light of social concern. Jack's instruction proposed to engage students in mathematical deconstructing media images and messages to examine underlying biases. Through various strategies, participants' students used argumentation to answer decidability and believability questions. Generally in the math classroom, we do not engage students in disassembling and mathematically understanding cultural or media messages and such concern, since it was not addressed by participants regular school mathematics, was addressed through locally developed materials supported either by data or discussion how relationships expressed mathematically through *Narratives* encouraged civic action.

Finally, how that civic action was shaped took on two forms: inductive and proletarian. Strategies organized by participants as inductive results of action highlighted traditional school mathematics knowledge such as recognizing and describing patterns and functions while participant pedagogy through proletarian results allowed for a more student-centered and socially oriented meaning. The importance to CML is that both contain the mathematics important for *framing* action and *taking* action. Through addressing each, at appropriate times, the CML educator values and models how mathematics can be expressed in relevant social engagement.

Bridging ontological spaces and CML cultural values. As a bridge between the ontological spaces of existence and participants' CML cultural values, teacher structures deal with the professionalization of teaching. The first structure, didactical structures, supported participants relations with other professionals who could understand demands from parents and school administrators, engage in professional development and dialogue, and address big ideas of the profession. Coping structures require us to deal with the physical and psychic demands of teaching: I am a parent, big sister/brother, counselor, public safety officer, community activist, and negotiator. All at the same time. Every day. Teacher structures help us work cooperatively, within our schools and within the profession, to transition beyond deploring our students to earnestly assessing our resources and capacities to best meet education goals. In the field, these structures are known by names such as professional learning communities or school-based learning teams.

As a bridge across spaces of existence and relevant knowledge, structures reflect our interaction with other educators while CML cultural values connect to what is valued and how we participate in classroom practices. The categorization of epistemological views into utilitarian, purist, and social change suggests participants needed to hold differing values at various times. Social and political pressures call for a utilitarian view while participating in resistance and change need a more activist, social change view. The contrasting views expressed in relation to the role of mathematical knowledge highlight existing challenges in preparing and retaining teachers. For example, as Owen commented how when he worked with two student teachers, they questioned why students were unable to work with fractions and had difficulty with derivatives. These comments exhibit the unrealistic view of new educators and also how unprepared they may be in meeting the needs of all students.

Limitations

Lastly, I would like to briefly address limitations of this research. While collecting data, one of the limitations experienced at Maria Montessori High School was the challenge imposed by the structure of the school's schedule. That is, students are on alternating week schedules, each week has different students and the content from the previous week gets retaught or repeated. This means that observations were of the same lesson but with a different group of students. The effect this had on observations was that I found myself writing similar observation notes but with different students.

Another limitation is that the nature of participants' instruction, when using sociopolitical topics, has potential to be devoid of mathematics. That is, in some instances it felt more like a political science, history, or social studies class. For example, discussion could get quite involved during issues related to social, economic, cultural, or political challenges in seeking a solution from mathematical means. I would suggest however that such discussion is necessary to set the context—reasoning about why there are a large number of sociopolitical narratives available piques student engagement to broader issues and contexts outside their classroom and immediate lives. In fact, there were points with all participants where it was hard to comment on the divorce of mathematics from class discussion. I believe this is because participants must accept multiple roles in being able to more actively discuss issues and focus on the mathematics later. That is to say, while a participant planned to address mathematics through an appropriate critical mathematics issue, discussion of the nature and scope of the issue might have potential to background the mathematics in a particular lesson.

Limitations relating to constructivism. In suggesting a conceptual framework rooted in the social constructivist view, there has been focused attention to the process of arriving at mathematical knowledge. This assumes both that learning and teaching are also parts of this knowledge. On the first assumption, that of learning as part of the process of constructing mathematical knowledge, needs to be addressed by offering details as to what an ontology for learners in such a view of mathematical knowledge. I have attempted to outline and justify one such ontology as the first and second spaces of existence. However, that does not dismiss other ontologies or even suggest what is outlined in this research are necessary to a critical mathematical literacy understanding. That is to say, should we adopt a different epistemological claim—say, that of a preexisting body of mathematical knowledge, as in the absolutist view—then supporting ontological approaches may be entirely different.

On the second assumption, that of teaching as part of the process of constructing mathematical knowledge, it is important to acknowledge that because this process is located in the social interaction between public and private realms, teaching changes and revises students assumptions as meaning forms in different realms. This meant that participants placed considerable value on the role of classroom discourse in establishing meaning. That is, supporting students with the materials and resources to challenge and critique in establishing meaning of being critically mathematically literate. This is seen in participants' largely aligning to a social change epistemology (see Table 9) in their reflections on the culture of participating in critical mathematical literacy. A similar study with different participants might not find such values or that participants might have different considerations for epistemological foundations for mathematical knowledge.

Finally, as with most ethnographic work, is the issue of time spent in the field. Since I was only at each school for a maximum of eight weeks, it was important to build participants', and that of their students', emotional valence (Paterson, 1994) by exhibiting that the relationship between the researcher and participant was equal in the classroom. While it was important for students to see that I was similar to their teacher in terms of status, knowledge, professional role within the school, etc., it was more important for the participant to feel similar status of trust in the research relationship. For each participant, overcoming emotional valence was a different process. With Owen and Gwen, it required participating in more before- and after-school activities such as staff meetings and conferences, in-services and district events. This was also the case with Jack, however since Jack was the department chair, it was also important for his colleagues only to see my role as classroom observer and not that of a potential new teacher; that is, since teaching positions at Jack's school were highly sought after, it was imperative that I would not be seen as a candidate for such position. This did not necessarily affect observations or data from the school but is important in acknowledging so that results and methods were consistent across all three participants.

Implications for continued research

Discussion and analysis of results suggest the means by which CML educators connect the permanent social world to students' temporary time in the classroom is a complex mix of ontological spaces that share a connected consciousness of the process of generating mathematical knowledge. Questions still remain as to the continued pursuit of individual trajectories within the described four spaces of existence. Likewise, considerations for fully integrating these into existing and future practices of professional development need to be identified and described.

Recall that the first ontological space involves the ways in which culture is transmitted. Three means were identified as practices in maintaining, revising, and mediating culture. This semantic relationship highlights ways critical mathematical educators make explicit traditions that maintain inequities and position their students to resist. As Gwen states: "…having some knowledge of [math] is used to keep people from being more educated." While this perspective was not fully elaborated upon, I would suggest that future research should continue to identify and describe means, resources, and instruction that permits students to resist mainstream culture and dominant mathematics.

In participating in a program of critical mathematical literacy, participants discussed the need to revise conceptions of mathematical knowledge as something to be handed down from teacher to students and to work with students to bridge individual meaning with public knowledge; administrators and parents need to support classroom teachers who have courage to step outside functional, utilitarian perspectives and encourage, active, transformative learning.

Implications for classroom practice

In thinking largely of implications for classroom practice, I would suggest that, as mathematics educators, we need to deeply reconsider the importance of our beliefs and continuing support of utilitarian and functional epistemological views in addition to how to more solidly prepare current and pre-service mathematics teachers with the professionalism needed to address the sociopolitical challenges of critical mathematical literacy teaching. Existing beliefs supporting a utilitarian view inform classroom practices that are not encompassing of the social and transformative nature of critical mathematical literacy and seemingly contra to positioning students to become active citizens and teachers as vehicles of change. An informed professional development, or pre-service didactics course, would address these themes.

Further, the professionalization of mathematics teaching as a tool for solely meeting needs of corporate and economic interests—that is, regarded primarily as a means for producing scientists and engineers—also does not honor the CML teacher's positionality to extend mathematical understanding to contemporary historical, cultural, or political problems that exist in society. Addressing the sociopolitical in the mathematics classroom extends traditional practice to increase mathematical discourse through involving contemporary civic concerns beyond the classroom (Skovsmose & Borba, 2004) in supporting dialogue and mathematical meaning that shift away from a utilitarian epistemology to one of social change; such as that exhibited in Appendices E-1 and E-2. In this way, if CML is to have stable outcomes and well-defined learning experiences, teachers need to consistently connect the sociopolitical with reflection on students' understanding with opportunities for transformative social change.

Conclusion

In the examination of how CML teachers conceptualize their practices and how those practices were demonstrated in the classroom, it is understandable that conceptualization involves considerations for an ontology of mathematics education that differs from what is historically accepted, that is, beginning with the Greeks, by positing a realm of ideals and being as existing independent of humanity; one where the process of doing mathematics is to gain access to this static Platonic realm. Here, the truth of mathematics is not in challenging or reexamining concepts and proofs, but in the beliefs in the axiomaticity of the conditions of numbers, functions, propositions, groups, and points (Priest, 1973).

In more recent times, mathematics education has transitioned across an unusual landscape of distortion of ontologically what is meant by mathematics, and more importantly, school mathematics, to our current historical crisis in which we increasingly "instrumentalize, professionalize, vocationalize, [and] corporatize" (Thomson, 2001, p. 244) expectations for education. The findings from this study suggest adopting a critical mathematical literacy perspective acknowledges that absolutist views are no longer the most ideal view; that teacher and student interaction, the questioning of what it means to do mathematics as an individual, as a citizen, as a community member, that the importance of indigenous and other subjugated ontologies (Semali & Kincheloe, 1999) replaces acceptance of solely one acceptable truth.

School mathematics is no longer valued for its absoluteness but is seen as fallible and malleable to human thought and action. Its role becomes one of enabling the practitioner to move through consumerist uses to more transformative ones. The school classroom, as a container for student and teacher discourse, encourages a literacy that is socially and politically aware and engaging. And, the position of the mathematics teacher is not to transmit knowledge or even to function as moderator between mathematical knowledge and application. Instead

A Survey of Sociopolitical, Instructional, and

Mathematical Knowledge Conditions That Have Shaped

the Critical Mathematical Literacy Movement

Abstract

This chapter orients the reader to mathematical literacy and then examines the sociopolitical, instructional, and mathematical knowledge conditions that help shape a CML perspective. Sociopolitical conditions highlight the need to reframe mathematics education for transformative individual and social change. Instructional conditions include the need to properly train mathematics teachers, and mathematical knowledge conditions highlight the need for revaluing content knowledge. The discussion reveals six domains for CML: (a) reflective capacities, (b) mathematical comme il faut, (c) mathematical fluidity, (d) mathematical prudence, (e) mathematical confidence, and (f) mathematical doubt.

Introduction

I would like to open by considering an issue with contemporary mathematics-teacher education. Consider that our work with pre- and in-service teachers, particularly what occurs in secondary mathematics and pedagogy methods classes, focuses on the authoritarian transmission, from teacher to student, of mathematical knowledge. More rarely are there mathematics educators who, through our work with education students and practicing teachers, promote mathematical instruction as opportunities to engage our high school students to actively participate in society with a mathematical sense of how personal and social change are possible.

This needs to change. As mathematics educators, we need to move past encouraging our education students and pre- and in-service teachers to continue the cycle of authoritarian and didactic models of mathematics instruction. What we need is to reconsider that our work as mathematics educators now needs to incorporate an understanding of culture and practices that support and empower mathematics teachers to consider possibilities of social change through mathematical literacy.

Early research largely defined mathematical literacy as mathematical knowledge and skills students need to participate in a technology and information-rich society (Denning, 1983). Educational researchers generally interpreted these skills as number sense and procedural fluency (to be characterized as advanced mathematical literacy) or as preparation for daily quantitative encounters (to be characterized as basic mathematical literacy). A third interpretation is that these mathematical skills must reflect the use of mathematical knowledge for personal and social change. This interpretation, the subject of this chapter, has been termed CML.

As Jablonka (2003) noted, in the past 10 years the results of domestic and international comparative assessments have generated increased interest in mathematical literacy. Further, developments in the professionalization of mathematical literacy as “achievement” by organizations such as the U.S. Department of Education and the International Association for the Evaluation of Educational Achievement have subtly prioritized an agenda of mathematical understanding as solely content areas and technical competency. Consider how, in the United States, the National Assessment of Educational Progress (NAEP) puts forth one frequently cited professional standard of mathematical literacy. Yet, its current framework for assessing mathematics understanding (National Assessment Governing Board, 2008) made no mention of mathematical or quantitative literacy as such, and, with content emphasis on number properties and operations, measurement, geometry, data analysis, and algebra, it is heavily weighted toward an unspoken acknowledgement of the primacy of procedural knowledge and technical fluency.

Another common comparative assessment is the Trends in International Mathematics and Science Study (TIMSS), which, like the NAEP, does not directly define mathematics literacy, but instead focuses on content areas (e.g., algebra, data and chance, geometry, and number) and cognitive domains (e.g., knowing, applying, and reasoning) that students use when they engage in mathematics content (Gonzales et al., 2001, 2008).

Finally, an international assessment that defines and balances the professionalization of mathematical literacy is the Program for International Student Assessment. Unlike NAEP or TIMSS, the Program for International Student Assessment considers the knowledge and competencies that are important for an individual’s socioeconomic and personal welfare. It boldly moves past defining mathematical literacy as solely procedural or technical competencies in content areas and suggests an interpretation of mathematical literacy as

> an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned, and reflective citizen. (Organization for Economic Cooperation and Development, 2006, p. 72)

Such professional standards of mathematical literacy suggest and frame it from a utilitarian, or functional, perspective: the content and cognitive focus of NAEP and TIMSS convey the image that mathematical understanding and literacy should be functional and not defined as empowering students with mathematical understanding to make connections between critical knowledge and possibilities for personal and social change. Viewing mathematical literacy solely from such a viewpoint does not entirely frame the different social spaces and contexts in which mathematical practices occur (Dowling, 1991). If we consider the social spaces for mathematical practices (e.g., academic sites, schools, work, or popular sites), valuing a utilitarian perspective means school mathematics is preparing mathematically literate individuals with knowledge for academic use (mathematical knowledge for the university) or work use (mathematical knowledge and techniques for work), but not for popular use (mathematical knowledge for daily and critical contexts). As mathematics educators, we need to recognize that mathematical discourse also happens in everyday and critical contexts. Understanding CML addresses the mathematical practices of social spaces and prepares individuals for engaged participation in democratic society.

If we examine the thoughts of teachers on CML, we see that the research suggests educators interpret it as knowledge and understanding that prepares students with a sense of how mathematics can interact with the world for transformative and social change (Frankenstein, 1983; Gutstein, 2003; Skovsmose, 1994). Yet, existing professional standards of mathematical literacy do not account for the mathematical discourse used for positive personal and social change.

As long as mathematics educators continue to support authoritarian or utilitarian modes of mathematical instruction and continue to accept existing professionalizations of mathematical literacy, the role of critical mathematical literacy will not gain. It may not be until we support our education students and teachers with knowledge and practice in pedagogy of social change through mathematics that we will begin to see wider educational and societal acceptance around instruction for CML, and through these actions empower teachers with practices needed to resist existing professional standards of mathematical literacy as functional and utilitarian.

Two Perspectives on Mathematical Literacy

In the 21st century, social concerns, both domestic and global, have encouraged researchers to question the value of what mathematics students are learning, giving rise to two competing schools of thought, each stressing a different understanding—AML and BML. AML is oriented toward understanding workplace encounters that involve mathematics, such as statistics in annual corporate reports or understanding sufficient to design optimal flight paths between cities. BML is designed to prepare students for the needs of their current and future lives (De Lange, 2001), the present and future needs of the individual; drifting away from considering mathematical knowledge independent of human thought and toward an analysis of an activity (Freudenthal, 1973) in which mathematical knowledge is discovered through the individual's activity.

Advanced mathematical literacy. Mathematical knowledge has recently been framed from the perspective of an emerging literacy, as mathematics educators begin to promote and accept multiple forms of mathematical knowledge (Tirosh, 1999). However, historically what is valued as mathematical knowledge has followed a logicist perspective that believes all mathematical knowledge is derived from the principles of logic (Shapiro, 2000), meaning that the role of mathematical knowledge is merely functional and that mathematical development and experience can be routinized and memorized. Such a perspective is tied to number sense (National Center for Education Statistics, 1993) and promotes instruction biased toward numerical fluency through emphasizing the understanding students need to participate in a technocratic information society (Mullis et al., 1998).

This early interpretation of mathematics literacy was supported by industry in that ultimate reasons for teaching mathematics to all students is because mathematics is primarily value in practical and scientific affairs (Carss, 1986). Educative intentions were “not to prepare students for university, but to introduce them to modern applications of mathematics in a technological society” (Howson et al., 1981, p. 173).

This type of mathematics education became self-selective; only those with an advanced mathematical understanding could continue to study (Moses & Cobb, 2001). Known as AML, it bridged instructing for mathematical knowledge and instructing for mathematics literacy, seen as necessary mathematical understanding for scientific and technical careers.

Basic mathematical literacy. While AML follows a logicist perspective, in BML we see adoption of process-oriented problem-solving methods(Lakatos, 1976; Polya, 1945; Popper, 1959) that interpret mathematical knowledge as more fallibilist in nature (Ernest, 1991). That is, allowing the individual more autonomy in investigating and establishing knowledge. This drift from a logicist genealogy interprets the role of mathematical knowledge as less absolute and more formalist, more organic. Mathematical knowledge was expanded to include quantitative practices (Denning, 1997), an understanding of the language of mathematics, and competencies individuals would need for present and potential needs. This latter emphasis on individual competencies was discussed at the end of the 19th century in the United States by the National Education Association (1899), which held that “there should be applications of algebra, geometry, and arithmetic to each other, and also to various sciences and the practical affairs of life” (p. 196).

This transition furthered the mathematics literacy perspective of early authors (De Lange, 2001; Steen, 1997) and promoted distinct characteristics for AML and BML. Generalized characteristics of AML and BML were discussed in OECD (2006) and De Lange (2001). These are organized in Table 10 to also incorporate secondary mathematics teachers’ viewpoints.

Table 12

Specific Characteristics of AML and BML

Advanced Mathematical Literacy	Basic Mathematical Literacy
Procedural fluency Ability to formulate, represent, and solve problems, and a disposition as a doer of mathematics (Kilpatrick, 2001)	Disposition, knowledge, and beliefs needed to engage in everyday quantitative situations (Statistics Canada & Organization for Economic Cooperation and Development, 2005)
Confidence to approach complex problems (Steen, 2001)	Contextual use of mathematics (Mathematics Council of the Alberta Teachers’ Association, 2005)
(a) Consciousness of what has been learned, (b) capacity for aesthetic appreciation, and (c) fluency with the language of mathematics (Mathematics Council of the Alberta Teachers’ Association, 2005)	Meaningful use of mathematics in ways that meet an individual’s life needs as a reflective citizen (Organization for Economic Cooperation and Development, 2006)
	Quantitative ability to understand commonplace issues (Steen, 2001)
	Ability to use mathematics in routine tasks, employment, and recreationally (Steen, 1997)

Note. Adapted from “A Brief Survey of Mathematical Literacy,” by Fish, 2008a, Mathematical Association of America Special Interest Group in Quantitative Literacy Newsletter, 1, 6–7.

Critical Mathematical Literacy

Hence, the two types of mathematics literacy handled the role of mathematical knowledge as functional (AML) or organic (BML), and considered the role of the individual to be interactive through mathematically passive means with their environment, and did not use their mathematical experiences for personal, social, or transformative change. CML considers the role of the individual to be active—that through individuals' understanding of mathematics they can exercise appropriate action based on the situation, requiring a more informed mathematical discourse to answer society's need to move beyond a traditional logicist perspective on mathematical understanding. Understanding CML considers the role of the individual, not explicitly acknowledged by the current understanding of AML and BML.

Considering the role of the individual as more active acknowledges nontraditional models of mathematics education that incorporate knowledge and understanding traditionally gained in a classroom setting, but also promote the role of students as more active in understanding their sociocultural experiences. These models of mathematical instruction may be project based (Barron et al., 1998; Jurow, 2005), situative (Lave et al., 1984), design based (Shaffer, 2005), or transformative (Frankenstein, 1983, 1990, 1994; Gutstein, 2006; Moses & Cobb, 2001; Skovsmose & Valero, 2001), this last of primary concern.

Several components of these models come together under CML. By promoting mathematics literacy as an interconnected lexicon of ideas based on human activity, CML becomes the knowledge and understanding that enables students to see how their mathematical knowledge can interact with a modern globalized society for positive and transformative social change. This suggests an interpretation of CML as informed mathematical discourse contextualized to account for the role of the individual and the situational role of mathematical knowledge (see Figure 15).

Constructivism lends itself as a tool for interpreting how teachers think about CML practices and the nature of their strategies for such instruction. This school of thought interprets mathematics instruction as a social activity. Mathematics education researchers who draw on social constructivist ideas, especially with the outcome of transformative change in mind, recognize the importance of sociopolitical issues in their instruction, interpret students as engaged participants in making mathematical knowledge, and see empowerment of their students as necessary to achieve change through mathematics.

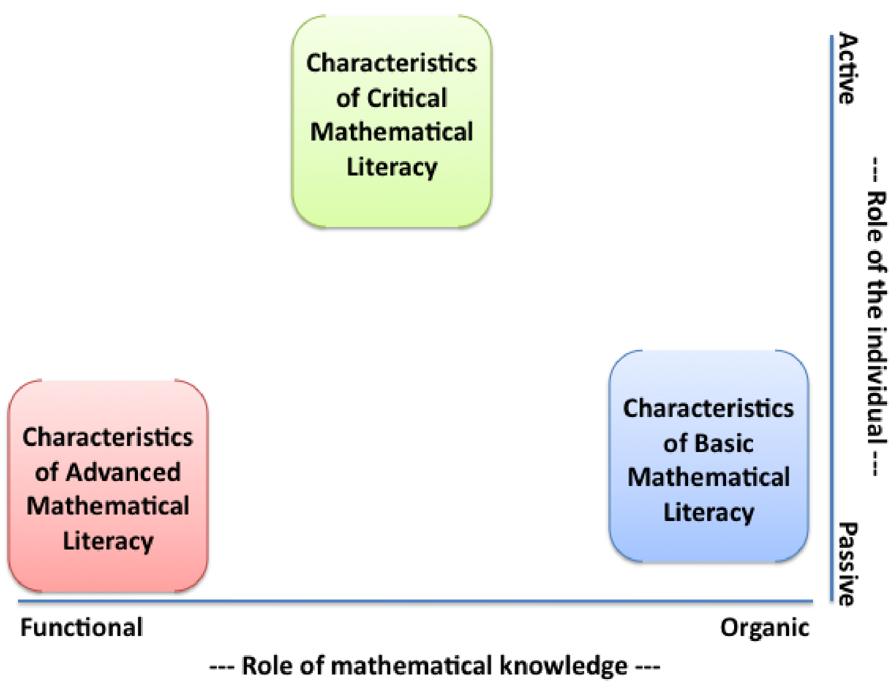


Figure 15. Reinterpreting mathematical literacy

Considering modern thought on the relationship between social constructivism and CML, we see the beginnings in the work of Peirce (1903), attempting to bridge the absolute truth of logicism with a more fallibilistic and public-inquiry approach to scientific knowledge. This shift from knowledge as existing a priori to public, debatable knowledge proposes, through knowledge construction, that evidence is an "objective factor inviting universal examination [and] … conceives of its results as essentially provisional or corrigible" (Buchler, 1940, p. x). Scientific knowledge is no longer perceived as infallible, but can be extended to account for the nature of mathematical knowledge as something beyond an absolute truth. Popper (1959) reckoned that knowledge is discoverable, not established a priori, through the logic of scientific discovery. In modern times, this method of scientific discovery, The Scientific Method, has become ingrained in the teaching of the biological and life sciences.

Mathematicians such as Polya (1945) acknowledged that mathematical knowledge is constructed through a social process: a problem is examined, acted on, and reflected upon at completion. Another strategy in the process of discovering mathematical knowledge is a four-stage process (Hadamard, 1954): preparation—"mobilizing ideas"; incubation—constructing various combinations of ideas; illumination—obtaining results; and conscious work—expressing results. But perhaps the most important model of social constructivist theories of mathematical knowledge is the logic of mathematical discovery proposed in seven stages by Lakatos (1976).

This need to account for situational mathematical understanding and an increased understanding of the role of the individual means educators must shift understanding of mathematics literacy from solely AML and BML, to account more for the role of the individual in the process of letting-something-be-seen in their understanding of mathematics literacy. This is what Ernest (1991) described as the social constructivist approach to mathematics education,

> in which … mathematical knowledge [is] corrigible and quasi-experimental … [and] knowledge is culture-bound, value-laden, interconnected and based on human activity and enquiry. [Where] both the genesis and the justification of knowledge are understood

> to be social, being located in human agreement … [allowing] knowledge, ethics, and social, political, and economic issues [to be] strongly inter-related. (p. 197)

Thus, in defining CML to include situational mathematical understanding, the relationship between individuals and the role of their mathematical understanding is sensitive to the particular situation. It is not a great leap to connect this to the 19th century concept of social progress, and the rise of modern science. From the Renaissance idea of "man as the measure of all things" and the Enlightenment emphasis on rationality came the concept that society is not necessarily static but could advance in social and political conditions over time. People could aspire to improve their lot, to achieve a measure of justice, and to have some influence over their political fate. CML flows from this intellectual tradition.

I reinterpret mathematics literacy to acknowledge AML and BML practices but also promote and encourage an understanding between the role of the individual and the situational role of mathematical knowledge (see Figure 1). This interpretation exhibits the nature of the relationship between individuals and their mathematical knowledge, with categories composed of factors or conditions of each aspect of mathematics literacy outlined in Table 1 for AML and BML, and highlighted below for CML.

Conditions of Critical Mathematical Literacy

Sociopolitical conditions accentuate the need to reframe mathematics education for transformative social change; instructional conditions include the need to properly prepare mathematics teachers. Mathematical knowledge conditions highlight valued content. The discussion proposes six domains for CML: *reflective capacities, mathematical comme il faut, mathematical fluidity, mathematical prudence, mathematical confidence,* and *mathematical doubt*.

Sociopolitical Conditions

A discussion of the sociopolitical conditions of CML follows classical interpretations of mathematics literacy that have framed the discussion of mathematical knowledge as largely functional and a form of content knowledge largely beneficial for technocratic (Denning, 1983) or workplace (Forman & Steen, 1994) applications. This kept the role of mathematical knowledge Platonic—an ideal realm independent of human thought—and valued the participation of the individual as a ritual of unquestioning memorization.

Neither AML nor BML addresses the use of mathematics for transformative individual and social change. The problem of dividing mathematics literacy into two schools of thought was that the mathematical content of AML focused on producing an educated workforce versed in algorithmic and procedural fluency, whereas the practical, everyday mathematical content of BML did not acknowledge other legitimate social spaces or mathematical practices needed by society.

Recognizing neglected uses of mathematics raises a concern for the social and political aspects of learning mathematics. The critical mathematics-education movement promotes access to mathematical ideas for all people, independent of race/ethnicity, gender, or class (Skovsmose & Borba, 2004) and dissolves class and racial boundaries to prepare every student for full participation in society. It means moving past algorithmic and procedural fluency to empower students with the skills and knowledge they need for a successful life, enabling them to understand biases, inconsistencies, and limitations contained in media graphical displays and political policy statements. Hence, in a new information and technology society, a new mathematics literacy is necessary for transformative social change (Frankenstein, 1983).

In response to society's call for an educated workforce, quantitative reasoning remained stubbornly divided into AML and BML. AML sought to produce efficient and productive workers who could apply their knowledge to workplace applications, whereas BML became the accepted knowledge society expected citizens to have, such as the ability to read a bus schedule or understand a car's gas gauge. In this stratified state, individuals' technical knowledge perpetuated established social and class positions because mathematics literacy was not made accessible to all. Illiteracy in mathematics was acceptable to Americans, whereas illiteracy in reading and writing was not (Moses & Cobb, 2001); that is, failure was tolerated in mathematics, but not in English-language studies. Because individuals need technical knowledge to succeed in most fields today, the acceptance of failure in mathematics has too often obscured socioeconomic realities (Frankenstein, 1983).

The function of mathematics education has been self-selecting. That is, as prior interpretations of mathematics literacy were biased toward AML and emphasized the needs of a technocratic industrial society, mathematics education aimed to identify potential mathematicians and direct them into mathematics, science, or technical majors after high school (Moses & Cobb, 2001). Apple (2000) referred to this as a thinning of morality, in which educators move away from "principles of the common good" (p. 251) to policies and practices in which "the competitive individual of the market dominates and social justice will somehow take care of itself" (p. 255). The CML movement arose partly in response, representing an effort to increase students' critical sociopolitical consciousness (Gutstein, 2008) and make classroom mathematics more empowering (Gutiérrez, 2002; D. B. Martin, 2000) and transformative (Frankenstein, 1990; Jablonka, 2003). For example, Frankenstein's work (1990) promoted CML by using statistical data to reveal how political and economic factors have denied minorities access to science, technology, engineering, and mathematics professions. Frankenstein's investigation, using real data, explored factors that limit access to mathematics knowledge and perpetuate American culture's complacency in developing a mathematically illiterate citizenry.

AML and BML foster specialized workers and citizens. AML produces productive, efficient workers, while BML produces mathematically literate individuals who can apply number sense to everyday problems. People may apply mathematical principles to their work or daily life, but are not necessarily able to ask what their greater role may be as consumers and users of mathematics. In contrast, students from CML classrooms are better able to grasp their roles and capabilities as holders of mathematical power. For example, students can mathematically analyze racial-profiling policies or the effects of gentrification (Gutstein, 2006). They begin to see the potential of mathematical knowledge to break down barriers of gender, class, and ethnicity (Thomas, 2001). CML enables students to understand how to make socially aware decisions and critically interpret information; they can use science and technology critically and responsibly and understand the cultural value of mathematics as a relevant language to be mastered (Skovsmose & Valero, 2001).

Instructional Conditions

Timothy and Quickenton (2005) found that preservice teachers were not well prepared to teach from a CML perspective. Preservice teachers found the curriculum unfamiliar when their existing notions of mathematics literacy were primarily aligned with an AML perspective. When pressed to develop students' CML, preservice teachers relied on strategies and justifications they had experienced as students. Gutstein (2006) posited that if mathematics educators are to successfully develop high school students' CML skills, teacher-education programs must prepare preservice teachers to deconstruct media images and representations and ask questions that future students should be taught to ask. New teachers develop mathematical confidence and doubt concurrently with their students, for example, in classrooms that reinvent mathematical ideas through the mathematization of realistic situations and problems (Gravemeijer, 1998). In classrooms that employ project-based instruction (Barron et al., 1998; Jurow, 2005), students examine realistic problems and participate in discourse in ways that reflect their pragmatic understanding and concerns, justifying why their perspective would solve a problem. Shaffer (2005) explored similar ideas in design-based instruction methods.

Arranged classrooms. In their work with CML educators, Skovsmose and Borba (2004) described mathematics classrooms as *current, imagined,* and *arranged* situations. When one peruses a day of the mathematics teacher's life, often what they capture is the current situation: students not bringing their calculators to class, incomplete homework, and student culture toward one another. The imagined classroom is just that: an imagined alternative to the current situation. This perusal in the day of a mathematics teacher captures an idealized improvement to the current situation—one in which students have access to calculators, homework is finished, teachers promote classroom dialogue, and student culture is supportive and accepting.

Arranged situations include the complexity of classroom situations and are "practical alternative[s] that emerge from a negotiation" (Skovsmose & Borba, 2004, p. 214) between classroom teachers, students, their parents, and administrators. Arranged situations include the idiosyncratic processes and procedures that are used by the classroom culture in handling its daily life. Students forget calculators, so the teacher loans them one in exchange for their ID card; the teacher does not notice the student hastily finishing homework as new material is presented; student culture becomes supportive of certain norms and biased against others.

Describing the CML classroom as an arranged environment can yield important interpretations about the instructional conditions through which CML emerges. The present situation, which in practice would be either an AML or BML classroom, needs to be reframed in pre-CML terms. The mathematics literacy that students develop in these classrooms is narrowly defined and not well suited for using mathematics critically. By contrast, the idealized classroom is an imagined situation in which a proficient teacher and the students acknowledge CML through dialogue and by dealing with realistic mathematical problems. An imagined classroom is described in the connected-knowing mathematics classroom model of Boaler and Greeno (2000). However, in contemporary public schools, the best alternative is the arranged situation, in which practical alternatives emerge through student–teacher negotiation, in which realistic mathematics permits students and teachers to develop mathematical confidence and doubt simultaneously.

In arranged classrooms, another role of the teacher is to develop students' reflective capacities, emphasizing the cultural value of mathematics as worthwhile and preparing students to mathematically understand socially relevant issues. By developing their reflective capacities, students cease to rely on their environment to create meaning and understanding, but instead provide their own interpretations.

Preparing students to mathematically understand social issues encourages the perspective that society will be better as a result of that understanding. When students exercise reflection about a situation, they transition from abstract or conceptual knowledge to the observed phenomenal experience of the abstraction. Frankenstein (1983) used data to encourage students to question established knowledge from a dialectical perspective, allowing students to make connections between critical knowledge and possibilities for personal transformation and social change.

Similarly, in CML, students carry life experiences and a sense of social awareness; thus, CML can give them "contexts, tools, and space to begin the complex process" of understanding their mathematical role in society (Gutstein, 2006). The teacher helps students translate, interpret, and understand that role. Just as a language teacher helps students interact with others who speak and understand the language, a CML teacher helps students interact with others who speak and understand mathematical language through representations, symbolism, and expressions.

Mathematical Knowledge Conditions

D'Ambrosio (1990) asserted that

> mathematics ought to prepare citizens so they cannot be manipulated and cheated by indices, so they can be allowed to change and to accept jobs which fulfill and appeal to their personal creativity [and which enable them to] be free to pursue personal and social fulfillment (p. 21)

This statement acknowledges the changing nature of mathematical knowledge needed for participation in an increasingly globally focused society. Here, mathematical knowledge is transformative in that its understanding supports students in pursuing social fulfillment. Students should understand when they are being mathematically misled or manipulated by others, either consciously or through incorrectly represented information. Students need to be critically aware and able to evaluate fundamental relationships between conceptual and procedural knowledge (Hiebert, 1986). For example, competence in the abstractness of algebra symbols, but lacking an understanding of how to use that knowledge to interpret a newspaper graph, means the student is deficient in procedural knowledge.

Innumeracy is the inability to handle numbers and data correctly or to mathematically evaluate everyday problems and solutions (De Lange, 2001; Paulos, 1988), leading to an overemphasis on computation and inadequate emphasis on mathematics literacy as a tool for transformation. For example, in the work of Mullis et al. (1998) the competencies for becoming numerate generally focused on mathematical knowledge for deconstructing career-related problems and situations, not on techniques for deconstructing natural, social, or political conditions and issues (D'Ambrosio, 1990). In contrast, mathematical knowledge taught from the perspective of mathematics literacy for transformation and social change (D'Ambrosio, 1990; Frankenstein, 1983, 1990, 1994; Gutstein, 2006), and the descriptive comparisons of Dossey (1997), guides individuals to positive mathematical participation in society by enabling them to justify critical interpretations of data and change their perceptions on social issues (Frankenstein, 1990).

CML can be classified as mathematical knowledge that helps students work in occupations that appeal to their personal creativity and "can offer a possibility for students to engage in a dialogue where critique and disagreement can emerge" (Skovsmose & Valero, 2001, p. 50). Students can become literate in the language of mathematics as the exploration of knowledge and its arrangement in possible contexts and applications becomes the mathematical knowledge that enables them to interpret everyday problems and occupational situations creatively.

Domains of Critical Mathematical Literacy

The first of the six descriptive categories for CML, reflective capacities, includes two parts: cultural values in mathematics, and reflective capacities for mathematically understanding social issues (Steen, 1997). Second, mathematical comme il faut is conceptual understanding and adaptive-reasoning capacities (Kilpatrick, 2001; Kilpatrick, Swafford, & Findell, 2001). These are the diverse concepts and methods used to "determine whether a solution is justifiable and how to justify it" (Kilpatrick et al., 2001, p. 131).

Third, mathematical fluidity is fluency with the language and skills necessary to evaluate others' mathematical thinking (Mathematics Council of the Alberta Teachers' Association, 2005); that is, learning the tools of mathematics, how to manipulate symbols, and how to solve equations. Fourth, mathematical prudence is the sense of how mathematics interacts with the world (Organization for Economic Cooperation and Development, 2006), including discourse and seeking social support through discussion of examples and counterexamples, enabling students to internalize concepts (Clarke, Emanuelson, Jablonka, & Ah Chee Mok, 2006) and increase control over their immediate environment (Zimmerman, 1989).

Fifth, mathematical confidence is the confidence to confront authorities and to think critically and mathematically as an individual (Steen, 2001). Finally, mathematical doubt includes preparedness to use mathematics to investigate and critique injustice and oppressive structures (Gutstein, 2006). Mathematical doubt best develops in shared learning environments in which dialogue with learning partners allows students to become more critically responsible and reflective about applications of the mathematical tools they have acquired (Totten, Sills, Digby, & Russ, 1991).

Reflective Capacities

Mathematical reflective capacities involve cultural values and mathematically understanding social issues (Steen, 1997). First, cultural values that pertain to mathematics reflect a positive belief that when mathematical understanding is applied to solve social problems, the community and society benefit. Society becomes a better place when individuals are involved in addressing relevant social problems, which results in positive change. Positive change is often stunted by cloudy debates that question the use of mathematics as a democratizing force (Schoenfeld, 2004).

The second part of reflective capacities concerns mathematically understanding social issues. Table 11 outlines several social issues and offers reflections on how educators may address these in their mathematics classrooms.

Table 11

Social Issues and the Reflective Capacities in Classrooms

Social Issue	Reflective capacities to mathematically understand social issues
Media-presented graphs	Are these representations mathematically valid?
Global warming	What are the mathematical criteria we may choose to determine genuineness?
Global financial crisis	Is the information politically based? In what ways do they agree or disagree with my mathematical understanding?
	Where is all the money going? Who has access or has lost access to it?
	Using mathematics to investigate the economic and statistical problems associated with developing a new fiscal architecture.
	Activities include mathematical modeling and simulations.

Mathematical Comme il Faut

Mathematical comme il faut involves conceptual understanding and adaptive reasoning capacities (Kilpatrick, 2001; Kilpatrick et al., 2001)—understanding relationships among individual facts (Hiebert & Lefevre, 1986), creativity, problem-solving skills, and mathematical intuition—skills mathematicians value as important. This is fundamental knowledge about how mathematics is organized and systematized. Although teachers wish to emphasize conceptual knowledge, they often instead emphasize procedural knowledge (Eisenhart et al., 2993).

Adaptive reasoning capacities (Kilpatrick, 2001; Kilpatrick et al., 2001) are "the capacity for logical thought and for reflection on, explanation of, and justification of mathematical arguments" (Kilpatrick, 2001). These two concepts together are organized into mathematical comme il faut*,* tools and habits of thought that prepare students to apply their CML at a high level (conceptual understanding) and to be secure in communicating and verifying why those arguments are valid (adaptive reasoning capacities).

Mathematical Fluidity

Mathematical fluidity is fluency with the language of mathematics and a capacity to evaluate the mathematical thinking of others (Mathematics Council of the Alberta Teachers' Association, 2005); a property of AML that CML has inherited. Under AML, fluency with the language of mathematics means mathematically literate individuals who are comfortable with expressions and symbols and have a complex understanding of the nature and order of operations. In the CML framework, *fluency* adds an individual's capacity to critically evaluate the mathematical thinking of others. For instance, when confronting data representations of global-warming trends, a critically, mathematically literate individual will comprehend patterns or equations, but will also critically question biases in the graph's axes, recognize the selectivity of the data, or engage in dialogue about countermathematical perspectives.

Mathematical Prudence

Mathematical prudence involves a sense of how mathematics interacts with the world and the understanding necessary for making well-founded judgments (Organization for Economic Cooperation and Development, 2006), and, from the CML perspective, an individual's skillful reason, caution, and ingenuity in the use of mathematical resources. Metaphorically, mathematical prudence can be visualized as a garden. The individual has spent time nourishing and developing resources and can exercise skill and ingenuity in selecting appropriate self-knowledge.

The CML “sense” is different from other forms of mathematics literacy. Under AML, this sense includes technical understanding and familiarity with procedures needed to get a correct answer. Under BML, this sense may involve interpreting bus and train timetables (for example, Organization for Economic Cooperation and Development, 2006) or comprehension about using geometric shapes to find area or volume. Under CML, this sense may involve knowing which tool(s) the activity requires.

The second component of mathematical prudence is the understanding necessary for making well-founded judgments (Organization for Economic Cooperation and Development, 2006). Applying mathematical prudence means effectively solving the problem and more confidently interpreting results and actions.

Mathematical Confidence

Mathematical confidence allows one to confront authorities and think critically and mathematically as an individual (Steen, 2001). Mathematical confidence is unique to CML, in that one of the goals of CML is to prepare students with the mathematical understanding necessary for transformative change. Mathematics educators regard mathematical confidence as important to apply and communicate in the mathematics literacy context, and to connect concepts to achieve results (Fish, 2008).

Examples of an individual’s use of mathematical confidence to confront authorities can be found in D’Ambrosio (1990), Frankenstein (1983), and more recently in Gutstein (2003) and Stocker (2008). What is notable about mathematical confidence is that it guides the development of students’ perception and mathematical knowledge in ways that prepare them to mathematically critique authority. Educators accomplish this by valuing students' lived experiences, using real-world data, and making connections between mathematics and positive (transformative) social change.

As the nature of CML is to prepare individuals mathematically for transformative change, this last component of mathematical confidence is empowerment to think critically and mathematically as an individual. Legal issues and power are often at the forefront of social and political discussion, as in the civil rights and women's movements: critically empowered citizens had broader perspectives on their participation in American society and culture. Empowerment through mathematical confidence means challenging media, cultural, and social representations, and analyzing them mathematically. For example, investigating the radiation–cancer link from cell-phone usage, examining mathematical reasons for or against gun ownership, or understanding adolescent sexuality and unwanted pregnancy, are moral dilemmas that can be represented through mathematical dialogue and investigation.

Mathematical Doubt

Mathematical doubt involves preparedness to use mathematics to investigate and critique injustice, and the skills necessary to challenge oppressive structures mathematically (Gutstein, 2006). Gutstein (2006) believed "students need to be prepared through their mathematics education to investigate and critique injustice, and to challenge, in words and actions, oppressive structures and acts" (p. 210). Preparedness means individuals have developed components of CML (reflective capacities, mathematical comme il faut, etc.), that enable them to apply mathematical doubt, when necessary, to information and social issues they encounter daily. The individual's mathematical understanding and sense of what is and is not possible support a healthy sense of mathematical doubt.

This preparedness gives the individual courage to investigate injustices and promote positive social change. Preparedness means there is an observable injustice that can be analyzed and potentially resolved through CML. For example, South African teachers use contexts of world population and AIDS to engage students with pattern recognition and pattern description, and students suggest ways these social problems may be understood or prevented (Sethole et al., 2006). Working with social problems by embedding them in mathematical tasks not only prepares students to investigate injustice, but enables them to mathematically challenge oppressive structures.

Putting Things Together

CML prepares students to explore their life situations through an understanding of how to make socially aware decisions that can lead to transformative social change. The CML model acknowledges that students can use mathematical knowledge critically and thus understand the cultural value of mathematics; because students recognize the value of realistic data, the model encourages them to question established knowledge and make connections between critical knowledge and possibilities for personal transformation and social change. Critical mathematics educators need to help students interact with others who speak and understand mathematical language through representations, symbols, and expressions. This perspective prepares citizens to preclude manipulation by, for instance, the media's mathematical misinformation, and empowers individuals to pursue employment options.

Finally, CML is evaluative in that it encourages students to be critically aware of fundamental relationships between conceptual and procedural knowledge; its application becomes a narrative, arranged with the contexts and applications of mathematical knowledge that enable individuals to interpret everyday problems and occupational situations.

Conclusion

This chapter examined how domestic and international comparative assessments primarily categorize mathematics literacy as being only AML, as in the case of NAEP and TIMSS, and in doing so do not acknowledge the important role mathematics plays in supporting individuals in working toward social change. Early critics, De Lange (2001) and Steen (1997), critiqued our understanding of mathematics literacy largely as AML. They recommended shifting from an understanding of mathematics as absolutist and functional to a more organic analysis of an activity (Freudenthal, 1973) in which knowledge is generated and discovered through an individual's mathematical activity.

AML could be understood as procedural fluency; an ability to formulate, represent, and solve mathematics problems; a disposition as a doer of mathematics; confidently approaching complex problems; conscious of what has been learned; able to appreciate the aesthetics of mathematics; and fluent with the language of mathematics. BML is understood as dispositions, knowledge, and beliefs needed to engage in everyday quantitative situations; contextual use of mathematics; meaningful use of mathematics to meet an individual's life needs as a reflective citizen; the quantitative ability to understand commonplace issues; and the ability to use mathematics in routine tasks, employment, and recreation.

Finally, CML involves reflective capacities (cultural values with regard to mathematics and reflective capacities for mathematically understanding social issues), mathematical comme il faut (conceptual understanding and adaptive-reasoning capacities), mathematical fluidity (fluency with the language of mathematics and capacity to evaluate the mathematical thinking of others), mathematical prudence (the sense of how mathematics interacts with the world and understanding necessary to make well-founded judgments), mathematical confidence

(confidence to confront authorities and empowerment to think critically and mathematically), and mathematical doubt (preparedness to use mathematics to investigate and critique injustice).

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Appendix A: Descriptive Observation Data Protocol.

1. Describe in detail all the places in the classroom.

2. Describe in detail all the objects and manipulatives used in the classroom.

3. What are the ways classroom actions are developed and engaged in?

4. Describe in detail what happens during classroom activities.

5. Describe in detail all the events taking place.

6. Describe in detail the time periods that take place.

7. Describe the people involved.

8. Describe in detail all the things people are trying to accomplish.

9. Describe in detail the emotions felt and expressed.

Appendix A-1: Exhibit of daily professional materials

Rubric for final summative presentation.

Choose a social issue

- Define the problem
- Explain possible solutions
- Explain the cost of solutions
- Use your own graphs and use your own words

Math

Level 4	Level 3
Explains graphs and numbers	Explains graphs and numbers
Explains meaning of graphs	Explains appearance of graphs
Understand what you present	Understand what you present
Created your own graphs	Use a different source for graphs

Creativity and Presentation

Level 4	Level 3
Captures interest	Organized
Stands out	Understandable
Organized	Sometimes interesting
Understandable	Reasonable timing (not too long or short)
Good timing	

Information

Level 4	Level 3
References from many sources	References from many sources
Everyone can understand info	Most people can understand info
You understand info	You understand info
Organized into sections	Information lists solutions
Information explains solutions	

Appendix B: First Interview—Elements of an Educational Ideology for Mathematics Interview Protocol.

Participant's Pedagogy

1. How would you describe the pedagogical strategies you use on a daily basis?
   a. How would you describe the focus of what you do?
      i. Do you focus on procedures? Pose problems?
   b. Could you describe what this focus looks like in the classroom?
2. What do you feel is important that students take away when they leave your classroom?
   a. Technical fluency? Skills for society? Knowledge for change?
   b. If these students were also interviewed, what do you think they would find most memorable about your classes?
3. By what means do you respond to student needs for a mathematical environment?
   a. For example, in your classroom do you give priority to the needs of society? Industry? The needs of the individual?
4. What do you think has had the biggest impact on your instructional strategies?

Primary Elements

5. How would you describe your overall personal mathematical epistemology?
   a. For example, would you acknowledge that mathematics should be transmitted from teacher to student?
   b. That mathematical knowledge is absolute? Separate from reality? Based in human activity?

6. In thinking about your instruction, how would you describe the set of moral values required to teach the way you do?
   a. Would you acknowledge the principles of egalitarianism? Democratic participation? Citizenship?
7. How would you describe your beliefs regarding the students you teach?
   a. Would you say they are "empty vessels" or "clay to be molded"?
8. How would you describe your beliefs about society?
   a. Is it a social hierarchy to be maintained? System of inequities in need of change?
9. How would you explain your beliefs about the goals of education? What roles does education play in the development of the individual? Of society?

Secondary Elements

10. I'd like you to think about your views on the goals of math education. How would you describe how mathematics education contributes to society?
    a. Can you talk about your aspirations and goals as a mathematics educator?
11. I'd like for you to describe your view on school mathematical knowledge. How do you see it as connected to other realms of knowledge? Its relationship to culture? To society? To life?
12. How would you describe your view on learning mathematics?
    a. By what means do you believe children need to engage with mathematics? Through dialogue? Investigating? Posing and solving problems?
13. Would you next discuss your views on teaching mathematics. For example, do you engage or expect student-student and student-teacher discussion?
    a. What does that normally look like?

b. If you use cooperative work, what does it look like in your classroom? Why this method of student work in contrast to other methods?

c. How would you pose a problem? Can you describe a recent example?

d. How would you engage students in mathematical critical thinking? Can you describe a recent example?

e. What kinds of materials and topics do you use?

i. Are they socially relevant? How?

ii. In what ways do these materials and topics promote social engagement? Student empowerment?

14. Could you describe measures and modes of assessment to gauge mathematical competence and achievement?

a. In what ways might these typically be used in your classroom or school?

b. What were the most important reasons for considering these assessments?

15. What kinds of resources do you use in your instruction?

a. Can you describe what kinds of “authentic materials” (newspapers, official statistics) you use?

b. Can you describe a lesson in which these materials were used?

16. How would you describe the way you interpret individual student’s mathematical ability?

a. Biological? Shaped by culture? By the social environment?

17. What role do you see the curriculum as having? For example, what historical, cultural, or geographical resources should it address?

a. What do you see as the role of mathematics in nonacademic contexts?

b. How would you describe the role of mathematics in the reproduction of social disadvantage?

Appendix C: Second Teacher Interview Protocol

Please answer the following questions to the best of your ability. If possible, provide, for each question, an answer from each of the following perspectives:

a. Your beliefs about the nature of mathematics, including the influence of your prior experiences with mathematics on your current beliefs.
b. Your beliefs about learning mathematics, including your beliefs about critical mathematical learning (CML).
c. Your beliefs about teaching mathematics.

Reflective Capacities:

1. ***Cultural values with regard to mathematics.*** How would you describe the ways these beliefs guide your perspective on the cultural and societal importance of your instruction that directs students to be mathematically aware of social, political, and economic and cultural issues?
2. ***Reflective capacities for mathematical understanding of social issues.*** How would you contrast your beliefs with those of non-CML teachers you may know or have experience with? For example, consider the differences that may exist when you are preparing students to address relevant social problems.

Mathematical Comme Il Faut:

3. ***Conceptual understanding.*** What role do your beliefs play in helping students to understand relationships among individual facts, creativity, problem solving, mathematical intuition, and other topics related to how mathematics is organized.

4. ***Adaptive reasoning capacity.*** Please describe how your beliefs influence the ways in which you attempt to provide students with opportunities for logical thought and reflection, explanation, and justification of mathematical arguments.

Mathematical Fluidity:

5. ***Fluency with the language of mathematics.*** Describe how your beliefs affect how you prepare students to be comfortable with the expressions, symbols, and language of mathematics?
6. ***Capacity to evaluate the mathematical thinking of others.*** Explain how you prepare students with skills to evaluate and think critically about the mathematical thinking and reasoning of others.

Mathematical Prudence:

7. ***Sense of how mathematics interacts with the world.*** Please explain how your mathematical beliefs influence the ways in which your instruction promotes students' understanding of how mathematics interacts with the world.
8. ***Ability to make well-founded judgments.*** Please explain how your beliefs have prepared you for engaging with students in ways that enable them to make well-founded judgments and interpret actions and results?

Mathematical Confidence:

9. ***The mathematical confidence to confront authorities.*** As a mathematics teacher, please describe how your mathematical beliefs influence your instruction and pedagogy related to empowering students to be able to confront authority figures mathematically.

10. ***Empowerment to think critically and mathematically as an individual.*** How would you describe how your beliefs influence the ways in which you prepare students to think critically and mathematically as individuals?

Mathematical Dubiosity:

11. ***Preparedness to use mathematics to investigate and critique injustice.*** How would you explain the ways in which your mathematical beliefs are related to your instruction in preparing students to use mathematics as a means for investigating and critiquing injustice?

12. ***Skills necessary to mathematically challenge oppressive structures.*** Please describe how your beliefs influence the means by which you teach students skills, ideas, and concepts for mathematically challenging oppressive structures.

Appendix D: Tuberculosis lesson

Too close for comfort[2]

Lesson 14	
Short description:	Tuberculosis is a disease that spreads through the air when people are in very close contact with each other. Many homeless shelters are often horribly overcrowded. If disease and overcrowding are keeping people on the streets, can we do better?
Specific mathematics:	
	• Measurement • Geo & spatial sense • Patterning/algebra • Data & probability
Specific social justice topics:	
	• Class/poverty • workplace • Civics/community

OPENING

"There are many shelters that do not meet the UN standards for refugee camps in terms of public health measures" (Tuberculosis Action Group, 2003).

Tuberculosis (TB) is a disease caused by the bacteria Mycobacterium tuberculosis. One third of the world's people are infected and each year two million people die from the disease. Tuberculosis does not infect people equally: people who live in poverty are more likely to contract the illness.

Homeless shelters are places where people can go to seek warmth, food, and a bed/sleeping mat for the night. They tend to be run by NGOs, funding by the government or private donation. Government cuts in funding, less affordable housing, increased powers to landlords to set the cost of rent and evict tenants and low minimum wages all contribute to homelessness and cause the number of people using shelters to increase (TAG, 2003).

Sometimes people who are homeless choose to sleep on the streets even though it is very cold and there are spaces in shelters. One reason is the possibility of getting sick from sleeping in very

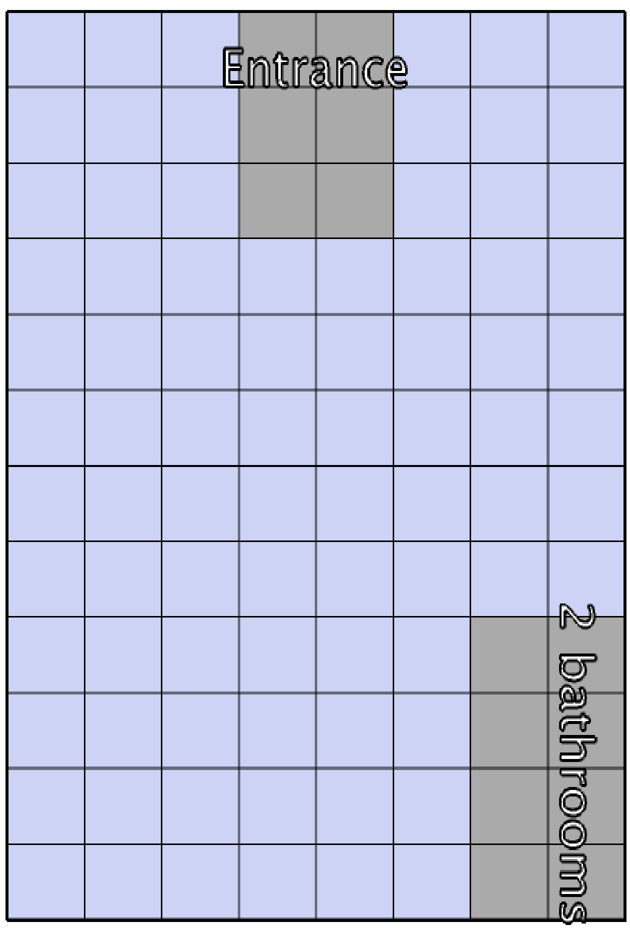
[2] Reprinted with permission from the publisher from Stocker, 2008, *Math that matters: A teacher resource for linking math and social justice*. Ottawa: Canadian Center for Policy Alternatives.

close quarters. The distance between sleeping mats can be as small as 36 CM and most shelters are consistently over 90% full.

--CAN YOU THINK OF OTHER POSSIBLE REASONS WHY PEOPLE MIGHT CHOOSE THE STREETS OVER A SHELTER?—

MATHEMATICAL UNDERSTANDING:

1. Use the floor plan of a sleeping space in a shelter to draw the number of sleeping mats you can fit in the space. Assume each mat is 2 meters long by 0.75 meters wide. Instead



3. Calculate the rate of people per square meter in this shelter. **NOTE:** The **UNITED NATIONS** sets the standard for space per person in refugee camps at 4.5 to 5 sq. meters per person. How does the rate for this shelter compare?

4. Calculate the rate of people per bathroom in this shelter.

5. Calculate the floor space of your living space in sq. meters.

6. Calculate the rate of people per sq. meter in your home.
7. Call a local real estate agent and ask them what the typical sq. footage is for a house in a wealthy area of town. Assume a family of four lives there. Calculate the rate of people per sq. meter in that house.

Part of the problem of shelters is what is called forced migration. In such programs, the shelter location is different each night of the week, so people must always be on the move. This is called transience. Other shelters have maximum stay lengths (two weeks, for example). The combination of close human contact and increase transience can make the spread of disease easier and tracking the disease more difficult (CSJ Foundation for Research and Education, 2003).

8. Imagine a person using the shelter system contracts influenza. Due to transience and crowding in shelters, examine the spread of the disease in the following table.

Time, in days	**Number of people infected**
0	1
1	2
2	4
3	8
4	
5	
6	
7	
8	
9	
10	

a. Complete the table. State the algebraic equation for this pattern.
b. Graph the results using time in days for the x-axis and the number of people infected for the y-axis. What kind of growth is this?
c. If the disease was really this contagious, and nothing were done to prevent its spread, how many days would it take for 10,000 people to be infected?
d. What kinds of things cause the spread of disease? What implications does this have for the shelter system?

SOCIAL ACTION:

- Have a shelter worker visit your class to talk about the system in your city.
- Visit the National Coalition on Housing and Homelessness (for CA teachers, USA teachers consider a similar organization) and sign the endorsement for housing solutions that work.
- View "Shelter in the storm" (Directed by Michael Connolly and "Street Nurse") from Toronto Disaster Relief Committee.

Appendix E-1: Exhibits of utilitarian epistemology and their critical mathematical literacy counterparts.

A-1. What could you do to engage your students so that they would learn to answer the following question?

1. A farmer's field is fenced on three sides with 1000 m of fence. What dimensions of fence give the greatest area of field?

B-1. The same math concepts can be taught with a social justice focus

1. How many people do you think could live comfortably in our classroom, if it was their only sleeping, eating, washroom, and living space?

Theresienstadt Concentration Camp was Nazi Germany's example that they used to show the world that they were not mistreating their Jewish people in 1944. It can be represented as a rectangular enclosure with buildings along one side and a barbed-wire fence along the other three sides.

2. With 1000 m of fence, what is the maximum area of the camp?

When the Nazis showed the world this "model" camp, there were 60, 000 people living there. Each person had about 2 m^2 of living space: that is the equivalent of 55 people living in a classroom, for years! This was the best that the Nazis had to show the world.

3. Where (if anywhere) do you think there are people living like this right now? What can you do about it?

Appendix E-2: Exhibits of utilitarian epistemology and their critical mathematical literacy counterparts.

A-2. What could you do to engage your students so that they would learn to answer the following question?

1. A rectangular prism has dimensions of 15.6 cm x 6.63 cm x 0.0109 cm. Determine the volume of the prism.

B-2. The same math concepts can be taught with a social justice focus

1. A US dollar has the approximate dimensions of 15.6 cm long x 6.63 cm wide x 0.0109 cm thick. Determine the volume of a US dollar.
2. Measure the classroom length, width and height. Determine the volume of the classroom.
3. How many US $1 bills would fit into the classroom?
4. The tobacco industry annual advertising budget is approximately 13.5 billion dollars. This does not include production, sales, wages, or research… just advertising. How many classrooms full of US $1 bills does this represent?
5. What can you do about this?

APPENDIX F: A brief discussion of teacher beliefs framing critical mathematical literacy

This appendix will discuss how mathematics teachers' beliefs are characterized and their relationship to classroom practice. An understanding of mathematics teachers' beliefs will frame the interpretation of their conceptualization of CML practices and instructional strategies. Several different models characterize teachers' insights and beliefs (Calderhead, 1996; Raymond, 1997; Thompson, 1992), factors that influence the relationship between the role of prior knowledge and beliefs in learning to teach (Borko & Putnam, 1996), or the relationship between the teacher and the curriculum materials (Remillard, 2005). In a discussion of teachers' beliefs and conceptions, Thompson (1992) cast a wide net regarding these belief systems, largely considered conceptions that include "beliefs, meanings, concepts, propositions, rules, mental images, preferences, and the like" (p. 130), stemming from Ernest's (1998) influences on mathematics teaching.

Calderhead's (1996) survey of teachers' beliefs and knowledge is not an appropriate model for interpreting teachers' beliefs because it takes a cognitive perspective that acknowledges it is "sometimes difficult to identify the distinguishing features of beliefs and how they are separated from knowledge" (p. 719). For this literature review I am seeking models of beliefs that emphasize knowledge as externalized and beliefs as internal, personal deductions. Calderhead interpreted teachers' knowledge and beliefs as "teachers' cognitions" (p. 715, p. 721), which does not seem to involve an understanding that teachers' beliefs, particularly mathematical beliefs, are influenced by prior subject-matter experiences.

Additional research on teacher beliefs suggested that they operate as filters through which teacher learning happens (Borko & Putnam, 1996) and that they influence decisions and actions made prior to, during, and after instruction (Philippou & Christou, 1997). Such perspectives treat beliefs as unconscious and do not seem to encourage an analysis of the idea that the action of a filter in regulating behavior is primarily unidirectional: it can act as a blocking mechanism by filtering a potential idea, and such perspectives do not enable teachers to reflect on their beliefs or actions.

Teacher knowledge is externalized, whereas beliefs are internal, personal deductions. In this distinction, knowledge and beliefs are separated. Knowledge reflects something that is learned—a conditioning that affects students' skills (Schifter, 2001)—but teachers' beliefs are about *what mathematics is* (Hersh, 1998) and are more related to their classroom instruction. Knowledge becomes sets of externalized truths to be manufactured or acquired, while beliefs are a personal, internalized ethos that regulates behavior and actions and provides space for reflection. Beliefs are internalized and personal in that they

> often include affective feelings and evaluations, vivid memories of personal experiences, and assumptions about the existence of entities and alternative worlds, all of which are simply not open to outside evaluation or critical examination in the same sense that … knowledge systems are. (Nespor, 1987, p. 321)

Consider, then, that teacher knowledge is externalized, accepted truths to be acquired, whereas teacher beliefs are personal conclusions. Hence, when answering the research question "What is the nature of strategies mathematics teachers use for instruction aligned with a CML philosophy?" teacher beliefs will be considered.

For this discussion I will build on Raymond's (1997) model of mathematics teachers' *mathematical beliefs*. These are the "personal judgments about mathematics formulated from experiences in mathematics, including beliefs about the nature of mathematics, about learning mathematics, and about teaching mathematics" (p. 552).

Mathematical Beliefs Formulated from Experiences in Mathematics

The literature suggests several influences on mathematics teachers' beliefs that are formulated from prior experiences in mathematics. The most conspicuous factor that influences teachers' mathematical beliefs stems from prior elementary (Charalambous, Philippou, & Kyriakides, 2002; Uusimaki & Nason, 2004) and secondary school experiences. Other factors include preservice teachers' field experience emphasizing procedural over conceptual knowledge (Eisenhart et al., 2993), and evidence suggesting the mathematical preparation of teachers as highly biased toward AML (McLeod & Huinker, 2007). Further, Carroll (1997) found that the personal experience of many elementary mathematics teachers was riddled with mathematics anxiety and negative attitudes toward mathematics. Interestingly, mathematical beliefs formulated from experience in teacher-education programs are minimally effective (Stipek, Givvin, Salmon, & MacGyvers, 2001), whereas preservice-methods course experiences are also characteristically negative for elementary education students (Mewborn, 2000) and tend to be utilitarian in approach (Britzman, 1986).

Mathematical beliefs about the nature of mathematics. On the nature of mathematics, Ernest (1991) proposed a model that characterizes teachers' mathematical beliefs about the nature of mathematics as utilitarian, purist, or social constructivist. This model is useful in the sense that it can be used in organizing thought around mathematical beliefs and views on mathematical knowledge from a mathematical literacy perspective, as outlined in Table 3. In the context of mathematical literacy, teachers with utilitarian beliefs focus on promoting functional understanding in their classrooms and see procedural and technical fluency as the basis for determining which knowledge and skills to value. These beliefs are embodied as a Platonic mathematical cultural system (Wilder, 1981) and have been previously discussed. Recall that this type of system regards knowledge as cumulative and considers attainment of mathematical knowledge as hereditarily transmitted, based on the needs of society/industry and not the individual. This perspective aligns with an AML ideology.

Mathematical beliefs recognized as purist view mathematics as being centered, and view their ultimate understanding as rooted in the mindset, practices, and everyday skills the individual needs to participate in society. The purist perspective aligns with BML. The social-change mathematical-beliefs perspective views the nature of mathematics as socially made (Wheeler, 1967, as cited in Ernest, 1991, p. 205), cultural (Bishop, 1988), and democratic knowledge (D'Ambrosio, 1990; Orey & Rosa, 2007; Skovsmose & Valero, 2001). This perspective aligns with CML. In Table 3, teachers' mathematical beliefs are related to beliefs about mathematical knowledge.

Mathematical beliefs about learning mathematics. Mathematical beliefs about learning mathematics can be summarized through the utilitarian/AML, purist/BML, and social-change/CML perspectives. Mathematics teachers whose mathematical beliefs about learning mathematics are utilitarian/AML perceive mathematics as functional and logically imitative, which leads to the perspective that learning mathematics should involve proficiency with mathematical tasks and rote practices, and that they are best learned through autocratic transmission from teacher to student. Further, evidence suggests that such beliefs about learning mathematics, at least among elementary mathematics teachers, are associated with the teacher's level of self-confidence and enjoyment of mathematics; that is, utilitarian teachers do not have as much enthusiasm (Stipek et al., 2001) as teachers who hold purist or social-change views.

Mathematics teachers whose mathematical beliefs are purist consider mathematical learning to be process oriented and more student centered than do those who adhere to the utilitarian/AML perspective. This perspective is encapsulated in the writings of the NCTM (2000, 2006), the Organization for Economic Co-operation and Development (2006), and the early work of Steen (1997). Learning occurs as students respond to their mathematical environment, and the mathematics teacher in turn responds to students' needs. Teachers with purist beliefs see learning as often being investigative or discovery-based, and teachers proactively engage in decision-making and appropriate methods to develop their students' mathematical learning.

Mathematical teachers' beliefs regarding learning mathematics from a social-change perspective are rooted in the writings of Freire (1970), Vygotsky (1978), and, more recently, Frankenstein (1990), Gutstein (2006), and Lee, Menkart, and Okazawa-Rey (2006). Supporters of social-change mathematical learning are similar to the purists (in that learning is process-based and student-centered), but they include the idea that mathematical learning and activity are actualized by the individual's culture (D'Ambrosio, 1990) and participation in daily society (Boaler & Greeno, 2000; Greeno, Collins, & Resnik, 1996). Mathematics teachers holding this perspective discuss learning mathematics for social change as more than functional learning (AML) and discovery-based (BML); the goal of learning mathematics for social change is to mathematically empower students (Steen, 2001) so that they can confront discriminatory and inequitable social situations (Gutstein, 2006) through mathematical means.

Mathematical beliefs about teaching mathematics. We have seen that mathematics teachers' beliefs formulated from experiences in mathematics are largely formed in prior school experiences and that perspectives on the nature of mathematics generally follow utilitarian/AML (Howson et al., 1981; McLeod & Huinker, 2007), purist/BML (De Lange, 2001; National Education Association, 1899), and social-change/CML (Frankenstein, 1983; Skovsmose, 1994) ideologies. This section will consider the role of mathematical beliefs about teaching mathematics. Such beliefs can be organized into *didactic* and *discussion-based* forms of teaching. As discussed in Boaler and Greeno (2000), didactic teaching is likely embraced by utilitarians and results in received forms of mathematical knowledge, while discussion-based teaching, likely embraced by purist and/or social-change teachers, results in connected or integrated forms of mathematical knowledge.

In line with the utilitarian perspective on instruction, which emphasizes functional and procedural fluency, the characteristics of mathematics teachers' beliefs on teaching mathematics are heavily content-based and largely professionalized by licensure examinations that value strong beliefs in the utilitarian perspective (Hill, Schilling, & Ball, 2004). A heavy professional emphasis on preparing these types of educators may be what transmits the utilitarian perspective from elementary teachers to elementary students. Even further, literature (Erlwanger, 1973) suggests that didactical teaching of mathematics does not require or promote deep mathematical understanding.

Conclusion

Recall that social constructivist perspectives on the nature of mathematics posit that it is socially encapsulated, and related beliefs about learning are that mathematics is learned through the individual's participation in social and cultural situations. From this stance, social constructivism borrows from von Glasersfeld's (1983) perspective that reality is constructed of experiences, and also the intuitionist idea (Dummett, 1999, 2000) that knowledge is created by humans to produce mathematical beliefs about teaching mathematics, which include

> genuine discussion, both student–student and student–teacher, … cooperative group work, project-work and problem-solving, for confidence, engagement and mastery; autonomous projects, exploration, problem posing and investigative work, for creativity,

> student self-direction and engagement through personal relevance; learner questioning of … contents, pedagogy and the modes of assessment used, for critical thinking; and socially relevant material, projects and topics, including race, gender and mathematics, for social engagement and empowerment. (Ernest, 1991, pp. 208–9)

The mathematics teacher engages in dialogical pedagogy and challenges and clarifies students' knowledge to incorporate other perspectives on mathematics. These beliefs about teaching mathematics "support a climate of critical questioning and scrutiny of mathematical arguments" (Ernest, 1998, p. 274).

APPENDIX G[3]: Gwen Cooper's statement against the closing of Heinz von Foester High School

Times change. Neighborhoods change. I grew up on Pied Walnut and Twin Bluff Streets. Half of the students who graduated from my elementary school went to Robin Heights, the other half to Heinz von Foester. I was in the Robin Heights group. In 1968, when I was a freshman, Robin Heights enrolled mostly white kids from blue collar families. I envied my Heinz von Foester friends. I thought it would be so much cooler to attend Heinz von Foester.

In the 1960's, Auburn was new. The housing was new and modern. The families who lived in the Heinz von Foester neighborhood looked modern and wealthy to me. Their houses were cute, their streets were clean. In June 1968, Robert Kennedy stopped to campaign in Auburn because he thought the people in the neighborhood mattered.

Times change. Neighborhoods change. I went away to university and returned in 1981. My old blue collar neighborhood was already falling on hard times. The people who stayed were mostly old and the people who moved in were underemployed. The struggle was becoming visibly harder for both the Robin Heights and Heinz von Foester neighborhoods. The cute modern houses were looking less cute.

I came to Heinz von Foester five years ago. I feel a great sense of urgency in my work. The students need my support and I don't just mean academically… psychologically. There are brilliant students at Heinz von Foester but they don't always believe they are. Kids who watch their siblings after and sometimes during school. Who don't have a quiet place to do homework, but they do it anyway. Who want to take piano lessons but they just aren't available. They are smart, resilient, funny, beautiful kids.

But, times change. Neighborhoods change. So, Heinz von Foester will likely close. I have one request. When you state justification for the closure, don't say it's about equity.

Equity to me and my students means equity of outcomes like more kids going to better universities, not equity of inputs as in more AP classes. Equity to me and my students means serious but gently run classrooms where diverse students can learn and teachers work to change their practice, where teachers engage in professional development like the [local professional opportunity for critical educators], Teaching for Social Justice, and Rethinking Schools. Equity to me means honoring the work of students and teachers over the past ten years in the area of small school reform wherever it has tried to flourish in this district. Equity to me is equity of worth and image that the growth and achievements of students in Heinz von Foester are just as normal as the achievements in any other neighborhoods. [Our] student's achievement isn't special because he or she is poor, or Latino, or an immigrant from China, or undocumented. It is

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special because that student is a hard working, dedicated high school student just like you find anywhere else.

If you vote to close Heinz von Foester, please embrace that the decision is political and economic. It’s not about equity. Equity is what we are providing now.